

A TEXT-BOOK
OF
COORDINATE GEOMETRY

[For B. A. & B.Sc. Part II and Degree Engineering Students]

PART II

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PREFACE

This book is Part II of Coordinate Geometry and is meant for B. A., B. Sc. and Degree Engineering Students of Indian Universities. Part I has been given a very warm reception by teachers as well as students and we earnestly hope that Part II will also meet their approval.

A new feature of the book is that 'Hints to Solutions' have been given at the end of the book.

We acknowledge our indebtedness to the authors of various standard works on the subject which we have freely consulted.

G. R. Inamdar
R. N. Jain

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CHAPTER I

USE OF DETERMINANTS IN ELEMENTARY PROBLEMS

1.1. Introduction. The determinants, which arise mainly in the solution of linear equations and in elimination, represent in a compact form many otherwise complicated expressions.

In this chapter some results in Coordinate Geometry will be expressed in an easily remembered form by making use of determinants.

1.2. Area of a Triangle. If $A (x_1, y_1)$, $B (x_2, y_2)$ and $C (x_3, y_3)$ are the rectangular coordinates of the vertices of a triangle, then

$$\text{Area } ABC = \frac{1}{2} \Sigma x_1 (y_2 - y_3) \quad [\S 1.5, \text{Part I}].$$

In the determinant form,

$$\text{Area } ABC = \frac{1}{2} \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \quad \dots \dots (1)$$

which is quite easy to remember.

The sign of the expression (1) is as explained in § 1.5, Part I.

Cor. The condition that the three points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) should be collinear is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0, \quad \dots \dots (2)$$

since the three points form a triangle of zero area.

Note. As in Part I, the axes are henceforth supposed to be rectangular unless the contrary is definitely stated.

1.3. Equation of the First Degree in x and y . By using determinants, we can easily show that the equation of a straight line is of the first degree in x and y and *conversely*, every equation of the first degree in x and y represents a straight line.

Let $A (x_1, y_1)$ and $B (x_2, y_2)$ be two fixed points on a given straight line and let $P (x, y)$ be a variable point on it.

Since the area of the triangle PAB is always zero, it follows that

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0, \quad \dots \dots (1)$$

which is the equation of the locus of P , i.e., the straight line AB .

Now (1) can be written as $x(y_1 - y_2) - y(x_1 - x_2) + x_1 y_2 - x_2 y_1 = 0$,
or as $ax + by + c = 0$,(2)

where a, b, c are constants.

Hence, the equation of a straight line is of the first degree in x and y .

Conversely, if three points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) lie on the locus (2), we have

$$\left. \begin{aligned} ax_1 + by_1 + c &= 0, \\ ax_2 + by_2 + c &= 0, \\ ax_3 + by_3 + c &= 0, \end{aligned} \right\} \text{.....(3)}$$

and

Eliminating a, b, c from the equations (3), we get

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0,$$

which shows that the three points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) are collinear, i. e., the locus (2), an equation of the first degree in x and y , represents a straight line.

1.4. Three Concurrent Straight Lines.

If three given straight lines

$$\left. \begin{aligned} a_1 x + b_1 y + c_1 &= 0, \\ a_2 x + b_2 y + c_2 &= 0, \\ a_3 x + b_3 y + c_3 &= 0, \end{aligned} \right\} \text{.....(1)}$$

and

all pass through the point (x_1, y_1) , then

$$\left. \begin{aligned} a_1 x_1 + b_1 y_1 + c_1 &= 0, \\ a_2 x_1 + b_2 y_1 + c_2 &= 0, \\ a_3 x_1 + b_3 y_1 + c_3 &= 0. \end{aligned} \right\} \text{.....(2)}$$

and

Eliminating x_1, y_1 from the equations (2), we get

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0,$$

which, therefore, is the condition that the lines (1) should be concurrent.

1.5. Necessary and Sufficient Condition that the General Equation of the Second Degree may represent a Pair of Straight Lines.

Necessary Condition. Let the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots\dots(1)$$

represent a pair of straight lines, then the expression on the left can be resolved into two linear factors,

$$\begin{aligned} \text{i. e., } ax^2 + 2hxy + by^2 + 2gx + 2fy + c &\equiv (lx + my + n)(l'x + m'y + n'), \\ \text{whence } &\left. \begin{aligned} ll' &= a, \quad mm' = b, \quad nn' = c, \quad lm' + l'm = 2h, \\ mn' + m'n &= 2f, \quad nl' + n'l = 2g. \end{aligned} \right\} \quad \dots\dots(2) \end{aligned}$$

$$\text{Now, } \begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix} = 0,$$

$$\text{i. e., } \begin{vmatrix} 2ll' & lm' + l'm & nl' + n'l \\ lm' + l'm & 2mm' & mn' + m'n \\ nl' + n'l & mn' + m'n & 2nn' \end{vmatrix} = 0,$$

$$\text{or } \begin{vmatrix} 2a & 2h & 2g \\ 2h & 2b & 2f \\ 2g & 2f & 2c \end{vmatrix} = 0, \quad \text{from (2)}$$

$$\text{i. e., } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0, \quad \dots\dots(3)$$

which is the required condition.

Sufficient Condition. If the condition (3) holds, it is possible to find x_1 and y_1 to satisfy the linearly dependent equations

$$\left. \begin{aligned} ax_1 + hy_1 + g &= 0, \\ hx_1 + by_1 + f &= 0, \\ gx_1 + fy_1 + c &= 0, \end{aligned} \right\} \quad \dots \quad \dots \quad (4)$$

and

Now transferring the origin to (x_1, y_1) and simplifying by using the relations (4), the equation (1) reduces to

$$aX^2 + 2hXY + bY^2 = 0,$$

which represents a pair of straight lines through the origin [§ 3.2, Part I].

Hence, the condition (3) is also sufficient.

1.6. Circle through Three Given Points. Let the equation of the circle passing through three given points (x_1, y_1) , (x_2, y_2) and (x_3, y_3) be

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad \dots\dots(1)$$

$$\text{then } x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0, \quad \dots\dots(2)$$

$$x_2^2 + y_2^2 + 2gx_2 + 2fy_2 + c = 0, \quad \dots\dots(3)$$

$$x_3^2 + y_3^2 + 2gx_3 + 2fy_3 + c = 0, \quad \dots\dots(4)$$

Eliminating g, f, c from equations (1), (2), (3), and (4), we get

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \end{vmatrix} = 0, \quad \dots\dots(5)$$

which is the required equation.

Cor. Four given points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) are concyclic, if

$$\begin{vmatrix} x_1^2 + y_1^2 & x_1 & y_1 & 1 \\ x_2^2 + y_2^2 & x_2 & y_2 & 1 \\ x_3^2 + y_3^2 & x_3 & y_3 & 1 \\ x_4^2 + y_4^2 & x_4 & y_4 & 1 \end{vmatrix} = 0. \quad \dots\dots(6)$$

1.7. Circle Orthogonal to Three Given Circles. Let

$$\text{the circle } x^2 + y^2 + 2gx + 2fy + c = 0 \quad \dots\dots(1)$$

cut each of the three given circles

$$x^2 + y^2 + 2g_r x + 2f_r y + c_r = 0 \quad (r=1, 2, 3),$$

orthogonally, then

$$\left. \begin{aligned} 2gg_1 + 2ff_1 - c_1 - c &= 0, \\ 2gg_2 + 2ff_2 - c_2 - c &= 0, \\ 2gg_3 + 2ff_3 - c_3 - c &= 0. \end{aligned} \right\} \quad \dots\dots(2)$$

and

Eliminating g, f and c from equations (1) and (2), we get

$$\begin{vmatrix} x^2 + y^2 & x & y & 1 \\ -c_1 & g_1 & f_1 & -1 \\ -c_2 & g_2 & f_2 & -1 \\ -c_3 & g_3 & f_3 & -1 \end{vmatrix} = 0, \quad \dots\dots(3)$$

which is the required equation

1.8. Solved Examples.

Ex. 1. If the three straight lines $a_1x + b_1y = 1$, $a_2x + b_2y = 1$ and $a_3x + b_3y = 1$ are concurrent, show that the three points (a_1, b_1) , (a_2, b_2) and (a_3, b_3) will be collinear.

If all the given straight lines pass through (x_1, y_1) ,

$$\text{we have } a_1x_1 + b_1y_1 - 1 = 0,$$

$$a_2x_1 + b_2y_1 - 1 = 0,$$

and

$$a_3x_1 + b_3y_1 - 1 = 0.$$

Eliminating x_1, y_1 , we get

$$\begin{vmatrix} a_1 & b_1 & -1 \\ a_2 & b_2 & -1 \\ a_3 & b_3 & -1 \end{vmatrix} = 0,$$

or

$$\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix} = 0,$$

which shows that the area of the triangle with vertices as (a_1, b_1) , (a_2, b_2) and (a_3, b_3) is zero, i. e., the three points (a_1, b_1) , (a_2, b_2) and (a_3, b_3) are collinear.

Ex. 2. Show that the circle circumscribing the triangle formed by the three lines $a_r x + b_r y + c_r = 0$ ($r=1, 2, 3$) has its centre on the x -axis, if

$$\begin{vmatrix} a_2 a_3 - b_2 b_3 & a_3 a_1 - b_3 b_1 & a_1 a_2 - b_1 b_2 \\ a_3 b_2 + a_2 b_3 & a_1 b_3 + a_3 b_1 & a_2 b_1 + a_1 b_2 \\ b_3 c_2 + b_2 c_3 & b_1 c_3 + b_3 c_1 & b_2 c_1 + b_1 c_2 \end{vmatrix} = 0.$$

The equation to the circumcircle is

$$\lambda_1 (a_2 x + b_2 y + c_2) (a_3 x + b_3 y + c_3) + \lambda_2 (a_3 x + b_3 y + c_3) (a_1 x + b_1 y + c_1) + \lambda_3 (a_1 x + b_1 y + c_1) (a_2 x + b_2 y + c_2) = 0, \dots\dots (1)$$

provided, the coefficient of x^2 = the coefficient of y^2 ,

and the coefficient of $x y = 0$,

$$\text{i. e., } (a_2 a_3 - b_2 b_3) \lambda_1 + (a_3 a_1 - b_3 b_1) \lambda_2 + (a_1 a_2 - b_1 b_2) \lambda_3 = 0, \dots\dots (2)$$

$$\text{and } (a_2 b_3 + a_3 b_2) \lambda_1 + (a_3 b_1 + a_1 b_3) \lambda_2 + (a_1 b_2 + a_2 b_1) \lambda_3 = 0, \dots\dots (3)$$

If the centre of the circumcircle lies on the x -axis, the coefficient of y in equation (1) is zero,

$$\text{i. e., } (b_2 c_3 + b_3 c_2) \lambda_1 + (b_3 c_1 + b_1 c_3) \lambda_2 + (b_1 c_2 + b_2 c_1) \lambda_3 = 0. \dots\dots (4)$$

Eliminating $\lambda_1, \lambda_2, \lambda_3$ from (2), (3) and (4), we get

$$\begin{vmatrix} a_2 a_3 - b_2 b_3 & a_3 a_1 - b_3 b_1 & a_1 a_2 - b_1 b_2 \\ a_2 b_3 + a_3 b_2 & a_3 b_1 + a_1 b_3 & a_1 b_2 + a_2 b_1 \\ b_2 c_3 + b_3 c_2 & b_3 c_1 + b_1 c_3 & b_1 c_2 + b_2 c_1 \end{vmatrix} = 0,$$

which is the required condition.

Ex. 3. If the four points in which the two circles

$$x^2 + y^2 + ax + by + c = 0, \dots\dots (1)$$

$$x^2 + y^2 + a'x + b'y + c' = 0 \dots\dots (2)$$

are intersected by the straight lines

$$Ax + By + C = 0, \dots\dots (3)$$

$$A'x + B'y + C' = 0 \quad \dots\dots(4)$$

respectively, lie on another circle, prove that

$$\begin{vmatrix} a-a' & b-b' & c-c' \\ A & B & C \\ A' & B' & C' \end{vmatrix} = 0.$$

(Allahabad '58 ; Delhi '59)

Method I. Any circle through the intersection of (1) and (3) is $x^2 + y^2 + ax + by + c + \lambda (Ax + By + C) = 0$, $\dots\dots(5)$
and any circle through the intersection of (2) and (4) is

$$x^2 + y^2 + a'x + b'y + c' + \mu (A'x + B'y + C') = 0, \quad \dots\dots(6)$$

where λ and μ are parameters.

If the four points of intersection are concyclic, then the circles (5) and (6) are identical for some values of λ and μ ;

$$\therefore a + \lambda A = a' + \mu A', \quad \text{or } a - a' + \lambda A - \mu A' = 0, \quad \dots\dots(7)$$

$$b + \lambda B = b' + \mu B', \quad \text{or } b - b' + \lambda B - \mu B' = 0, \quad \dots\dots(8)$$

$$\text{and } c + \lambda C = c' + \mu C', \quad \text{or } c - c' + \lambda C - \mu C' = 0. \quad \dots\dots(9)$$

Eliminating λ, μ from (7), (8) and (9), we get

$$\begin{vmatrix} a-a' & A & A' \\ b-b' & B & B' \\ c-c' & C & C' \end{vmatrix} = 0,$$

which is the required condition.

Method II. If the four points of intersection lie on a circle S , then the radical axis of (1) and (2), i. e.,

$$(a-a')x + (b-b')y + c-c' = 0, \quad \dots\dots(10)$$

the radical axis of S and (1), i. e., the line (3) and the radical axis of S and (2), i. e., the line (4) are concurrent.

Hence, eliminating x, y from (3), (4) and (10), we get the required condition.

Exercise 1

1. Prove that the area of a triangle inscribed in a parabola is twice the area of the triangle formed by the tangents at the vertices. (Agra 1955)

2. Prove that the area of the triangle formed by the normals to the parabola $y^2 = 4ax$ at the points ' t_1 ', ' t_2 ', ' t_3 ', is $\frac{1}{2}a^2 (t_2 - t_3) (t_3 - t_1) (t_1 - t_2) (t_1 + t_2 + t_3)^3$. (Agra 1957)

3. Show that the equations of the st. lines which pass

through (x', y') and are inclined at an angle θ with the line $ax + by + c = 0$ are

$$\begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ a \sin \theta \pm b \cos \theta & b \sin \theta \mp a \cos \theta & 0 \end{vmatrix} = 0.$$

4. Prove that the equation of the circle circumscribing the triangle formed by the lines $a_r x + b_r y + c_r = 0$ ($r = 1, 2, 3$) is

$$\begin{vmatrix} \frac{a_1^2 + b_1^2}{a_1 x + b_1 y + c_1} & a_1 & b_1 \\ \frac{a_2^2 + b_2^2}{a_2 x + b_2 y + c_2} & a_2 & b_2 \\ \frac{a_3^2 + b_3^2}{a_3 x + b_3 y + c_3} & a_3 & b_3 \end{vmatrix} = 0. \quad (\text{Agra 1952})$$

5. The cartesian coordinates (x, y) of a point on a curve are given by $x : y : 1 = t^3 : t^2 - 3 : t - 1$, where t is a parameter. Show that the points given by $t = a, b, c$ are collinear, if

$$abc - (bc + ca + ab) + 3(a + b + c) = 0. \quad (\text{Maths Tripos 1946})$$

6. Find the equation of the circle to which the triangle whose vertices are $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ is self-conjugate.

7. Prove that the circum-circle of the triangle formed by the lines $x \cos a_r + y \sin a_r = a \sec a_r + b \sin a_r$, ($r = 1, 2, 3$) passes through the point $(0, b)$.

8. Show that the general equation of all circles cutting at right angles the circles represented by

$$x^2 + y^2 - 2a_1 x - 2b_1 y + c_1 = 0,$$

$$x^2 + y^2 - 2a_2 x - 2b_2 y + c_2 = 0$$

is

$$\begin{vmatrix} x^2 + y^2 & x & y \\ c_1 & a_1 & b_1 \\ c_2 & a_2 & b_2 \end{vmatrix} + k \begin{vmatrix} x & y & 1 \\ a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \end{vmatrix} = 0,$$

where k is a parameter.

Answers

6.

$$\begin{vmatrix} x^2 + y^2 & 2x & 2y & 1 \\ x_1 x_2 + y_1 y_2 & x_1 + x_2 & y_1 + y_2 & 1 \\ x_2 x_3 + y_2 y_3 & x_2 + x_3 & y_2 + y_3 & 1 \\ x_3 x_1 + y_3 y_1 & x_3 + x_1 & y_3 + y_1 & 1 \end{vmatrix} = 0.$$

CHAPTER II

JOINT EQUATIONS OF STRAIGHT LINES COAXAL CIRCLES

2.1. Joint Equations of Straight Lines. It has been shown in § 3.1, Part I that the equation

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2) = 0 \quad \dots\dots(1)$$

is the joint equation of the pair of straight lines given by

$$A_1x + B_1y + C_1 = 0,$$

and

$$A_2x + B_2y + C_2 = 0.$$

Similarly, it can be shown that the equation

$$(A_1x + B_1y + C_1)(A_2x + B_2y + C_2)\dots(A_nx + B_ny + C_n) = 0$$

is the joint equation of the n straight lines given by

$$A_1x + B_1y + C_1 = 0, A_2x + B_2y + C_2 = 0, \dots\dots, A_nx + B_ny + C_n = 0.$$

Further, as already explained in § 3.2, Part I, the homogeneous equation of the n th degree

$$a_0x^n + a_1x^{n-1}y + a_2x^{n-2}y^2 + \dots\dots + a_ny^n = 0$$

is the joint equation of n straight lines, each passing through the origin.

Below we give some solved examples on joint equations of two or more straight lines.

Ex. 1. Show that the equation

$$a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 = 0$$

represents two pairs of straight lines at right angles, and that, if $2b^2 = a^2 + 3ac$, the two pairs will coincide.

(Agra '49 ; Allahabad '55)

Since the given equation is a homogeneous equation of the fourth degree in x and y , it represents four straight lines each passing through the origin. These four straight lines can be combined into two pairs of lines.

Let $a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2$

$$\equiv (a_1x^2 + 2h_1xy + b_1y^2)(a_2x^2 + 2h_2xy + b_2y^2) = 0, \dots\dots(1)$$

then equating coefficients, we have

$$a_1a_2 = b_1b_2 = a, \text{ i. e., } a_2 = \frac{a}{a_1} \text{ and } b_2 = \frac{a}{b_1}; \quad \dots\dots(2)$$

$a_1h_2 + a_2h_1 = -2b$ and $b_1h_2 + b_2h_1 = 2b$, (3)
whence $(a_1 + b_1)h_2 + (a_2 + b_2)h_1 = 0$ [by adding the equations (3)];

or
$$h_2 = -\frac{a_2 + b_2}{a_1 + b_1}h_1$$

$$= -\frac{a\left(\frac{1}{a_1} + \frac{1}{b_1}\right)}{a_1 + b_1}h_1, \text{ from (2)}$$

$$= -\frac{ah_1}{a_1b_1}. \text{(4)}$$

From (2), and (4), we see that the equation to the second pair of lines can be written as

$$\frac{a}{a_1}x^2 - 2\frac{ah_1}{a_1b_1}xy + \frac{a}{b_1}y^2 = 0,$$

or
$$b_1x^2 - 2h_1xy + a_1y^2 = 0, \text{(5)}$$

which represents a pair of st. lines at right angles to the first pair, i. e., $a_1x^2 + 2h_1xy + b_1y^2 = 0$(6)

Hence, the equation (1) represents two pairs of straight lines at right angles.

The two pairs will coincide, if each contains a right angle, i. e., if $a_1 + b_1 = 0$, when (5) also reduces to the form (6);

$$\begin{aligned} \therefore & a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 \\ & \equiv (a_1x^2 + 2h_1xy + b_1y^2)^2 \\ & \equiv (a_1x^2 + 2h_1xy - a_1y^2)^2 \quad [\because a_1 + b_1 = 0]. \end{aligned}$$

Equating the coefficients, we have

$$a_1^2 = a, a_1h_1 = b, \text{ and } -2a_1^2 + 4h_1^2 = 6c,$$

whence eliminating a_1 and h_1 , we get

$$-2a + 4\frac{b^2}{a} = 6c,$$

or $2b^2 = a^2 + 3ac$, which is the required condition for coincident pairs.

Note. If the equations (6) and (5) represent the pairs OA, OB and OC, OD respectively (fig.1), then OC is perpendicular to OA and OD is perpendicular to OB . By combining the four straight lines in the pairs OA, OC and OB, OD , we see that each contains a right angle. Hence, we can also prove the result above by showing

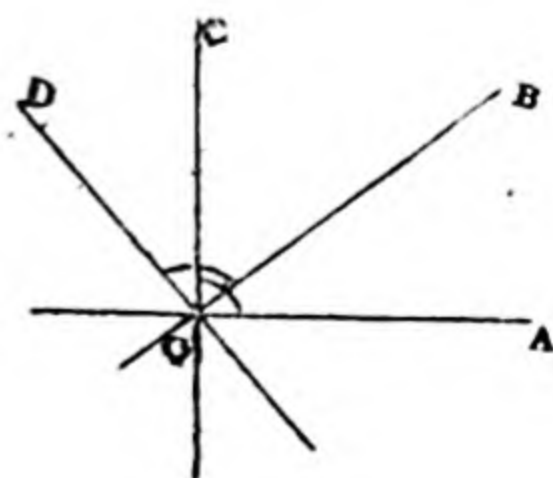


Fig. 1

that the equation (1) represents two pairs of st. lines each containing a right angle.

However, if we suppose that

$$a(x^4 + y^4) - 4bxy(x^2 - y^2) + 6cx^2y^2 \\ \equiv a(x^2 + \lambda xy - y^2)(x^2 + \mu xy - y^2),$$

where each of the factors on the right equated to zero represents a pair of lines containing a right angle, it is not always possible to show that λ and μ have real values.

Thus, the proof given above should be preferred.

Ex. 2. Show that the orthocentre of the triangle formed by the straight lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$ is a point (x', y') such that

$$\frac{x'}{l} = \frac{y'}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}. \quad (\text{Rajputana 1941})$$

Let $ax^2 + 2hxy + by^2 = 0$ represent the st. lines

$$y - m_1x = 0 \quad \text{and} \quad y - m_2x = 0,$$

i. e., $ax^2 + 2hxy + by^2 \equiv b(y - m_1x)(y - m_2x),$

then $m_1 + m_2 = -2h/b$, and $m_1m_2 = a/b$ (1)

The points of intersection of $lx + my = 1$ with $y = m_1x$ and $y = m_2x$ are respectively

$$A\left(\frac{1}{l + mm_1}, \frac{m_1}{l + mm_1}\right) \text{ and } B\left(\frac{1}{l + mm_2}, \frac{m_2}{l + mm_2}\right).$$

The equation of the st. line through $(0, 0)$ and perpendicular to $lx + my = 1$ is

$$mx - ly = 0, \quad \text{or} \quad \frac{x}{l} = \frac{y}{m} = \lambda, \text{ say,} \quad \dots\dots (2)$$

so that $(\lambda l, \lambda m)$ is any point on this line.

The equation of the st. line through A and perpendicular to $y = m_2x$ is

$$y - \frac{m_1}{l + mm_1} = -\frac{1}{m_2} \left(x - \frac{1}{l + mm_1} \right),$$

or $(x + m_2y)(l + mm_1) = 1 + m_1m_2$ (3)

Now, the orthocentre of the triangle OAB (where O is the origin) is the point of intersection of the lines (2) and (3). Hence, if $(\lambda l, \lambda m)$ is the orthocentre, then

$$\lambda(l + mm_2)(l + mm_1) = 1 + m_1m_2 \quad [\because (\lambda l, \lambda m) \text{ lies on (3)}],$$

or

$$\lambda = \frac{1 + a/b}{l^2 + m \cdot \frac{-2h}{b} + m^2 \cdot \frac{a}{b}}$$

$$= \frac{a+b}{am^2 - 2hlm + bl^2},$$

i. e., the orthocentre is the point (x', y') such that

$$\frac{x'}{l} = \frac{y'}{m} = \frac{a+b}{am^2 - 2hlm + bl^2}.$$

Ex. 3. Show that the lines represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

form a rhombus with the lines $ax^2 + 2hxy + by^2 = 0$, if

$$(a-b)fg + h(f^2 - g^2) = 0. \quad (\text{Osmania 1957})$$

The equation

$$S_1 \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots\dots(1)$$

represents a pair of st. lines

ABM , CBL respectively parallel to the st. lines OA , OC (Fig. 2)

represented by the equation

$$S_2 \equiv ax^2 + 2hxy + by^2 = 0, \quad \dots(2)$$

i. e., the figure $OABC$ is a parallelogram.

B , the point of intersection of the lines (1) is

$$\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2} \right)$$

[see § 3.5 (10), Part I] ;

\therefore the slope of the line $OB = \frac{gh - af}{hf - bg}$.

$$\text{The equation } S_1 - S_2 \equiv 2gx + 2fy + c = 0 \quad \dots\dots(3)$$

is satisfied by the points common to the line-pairs (1) and (2), and since it is linear in x and y it follows that the equation (3) represents the st. line AC .

Now, $OABC$ is a rhombus if OB is perpendicular to AC ,

i. e., if $\frac{gh - af}{hf - bg} \times \frac{-g}{f} = -1$ [\because the slope of (3) is $-g/f$],

$$\text{or } g^2h - afg = hf^2 - bfg,$$

$$\text{or } (a-b)fg + h(f^2 - g^2) = 0. \quad \dots\dots(4)$$

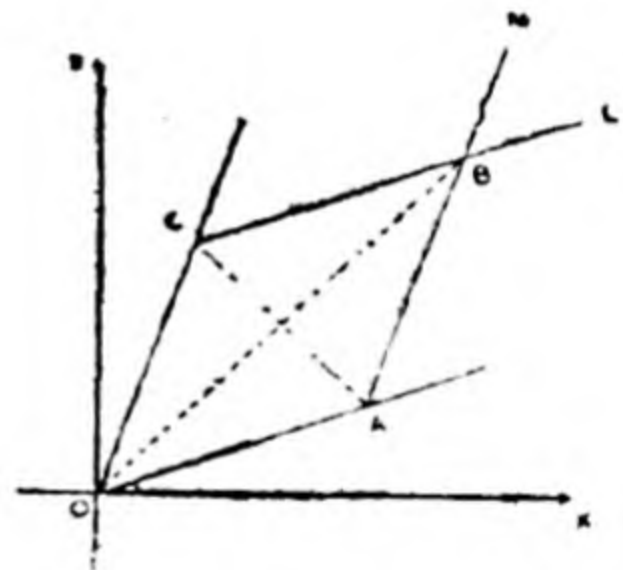


Fig. 2

Note 1. When $OABC$ is a rhombus, OB bisects the angle AOC . Hence, the problem above can also be stated as :

Show that one of the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$ will pass through the point of intersection of the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, if $(a-b)fg + h(f^2 - g^2) = 0$.

Note 2. When OB bisects \hat{AOC} , the two lines represented by (1) are equidistant from the origin. Hence, the problem above can also be stated as :

Show that the lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ are equidistant from the origin, if $(a-b)fg + h(f^2 - g^2) = 0$.

Note 3. The condition (4) can easily be shown to be equivalent

to $f^4 - g^4 = c(bf^2 - ag^2)$ (Rajputna '53 ; Agra '58)

This is left as an exercise for the reader.

2.2. Coaxal Circles. A system of circles in which each pair of circles has the same radical axis is called a *coaxal system*. (The word 'coaxial' is used by some writers.)

To find the equation, in the simplest form, of a system of coaxal circles.

Since each pair of circles of the system has the same radical axis, the centres of all circles of a coaxal system lie in one straight line perpendicular to the common radical axis [§ 5.41, Cor. 1, Part I].

Hence, taking the line of centres and the radical axis as the x -axis and y -axis respectively, the equation of any member of a coaxal system is of the form [§ 5.6, Part I]

$$x^2 + y^2 + 2gx + c = 0, \quad \dots\dots(1)$$

where g is a parameter and c a constant.

Note. If the line of centres of a coaxal system of circles be taken as the y -axis and the common radical axis as the x -axis, the equation representing the system is

$$x^2 + y^2 + 2fy + c = 0, \quad \dots\dots(2)$$

where f is a parameter and c a constant.

Particular Cases.

I. If the equations of two circles be $S_1 = 0$ and $S_2 = 0$, then

$$S_1 + \lambda S_2 = 0, \quad \dots\dots(3)$$

where λ is a parameter, represents a system of circles such that every pair has the same radical axis [§ 5.42 II, Part I]. Hence, the equation (3) represents a system of circles coaxal with S_1 and S_2 .

II. If $S=0$ and $L=0$ be the equations of a circle and a straight line respectively, the equation $S+\lambda L=0$, where λ is a parameter, represents a coaxal system of circles since every pair has the radical axis $L=0$ (§ 5.42 III, Part I).

2.3. Two Kinds of Coaxal Systems. Let a coaxal system of circles be represented by the equation

$$x^2 + y^2 + 2gx + c = 0, \quad \dots\dots(1)$$

where g is a parameter and c a constant.

The y -axis meets the circle (1) in points whose y -coordinates are given by

$$y^2 + c = 0, \quad \text{or} \quad y = \pm\sqrt{-c}; \quad \dots\dots(1)$$

hence, y -axis meets the circles (1) in real points (different or coincident) if $c \leq 0$ and in imaginary points if $c > 0$.

Again, the radii of the circles (1) are given by $\sqrt{g^2 - c}$. If $g = \pm\sqrt{c}$, the two members of the system (1) corresponding to these values of g have zero radii, *i. e.*, are *point circles*. The centres of these point circles, *i. e.*, $(\pm\sqrt{c}, 0)$ are known as **limiting points** of the system (1). These limiting points, which lie on the line of centres, are real (different or coincident) if $c \geq 0$ and imaginary if $c < 0$.

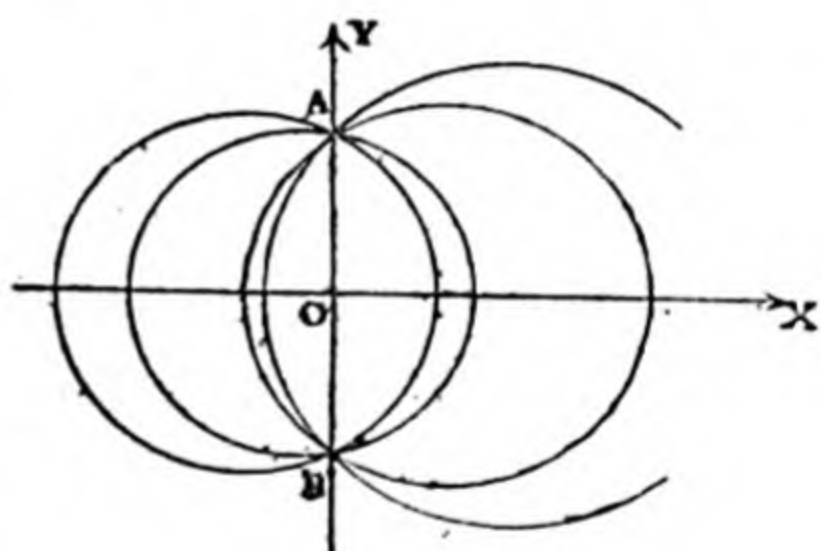


Fig. 3

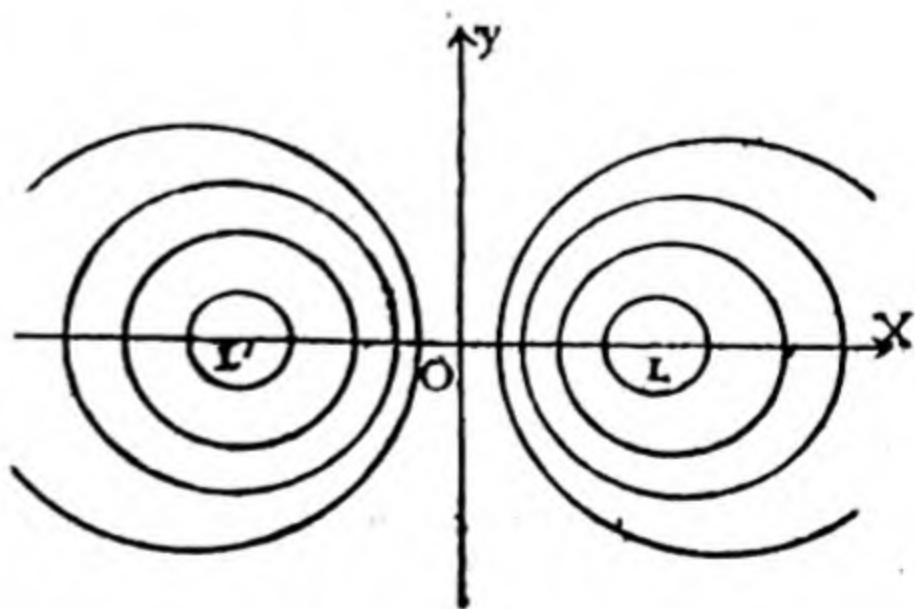


Fig. 4

Hence, we have the following cases :

I. If $c < 0$, all the circles (1) meet the y -axis in the same two points $A(0, \sqrt{-c})$ and $B(0, -\sqrt{-c})$ [fig. 3], but the limiting points are imaginary.

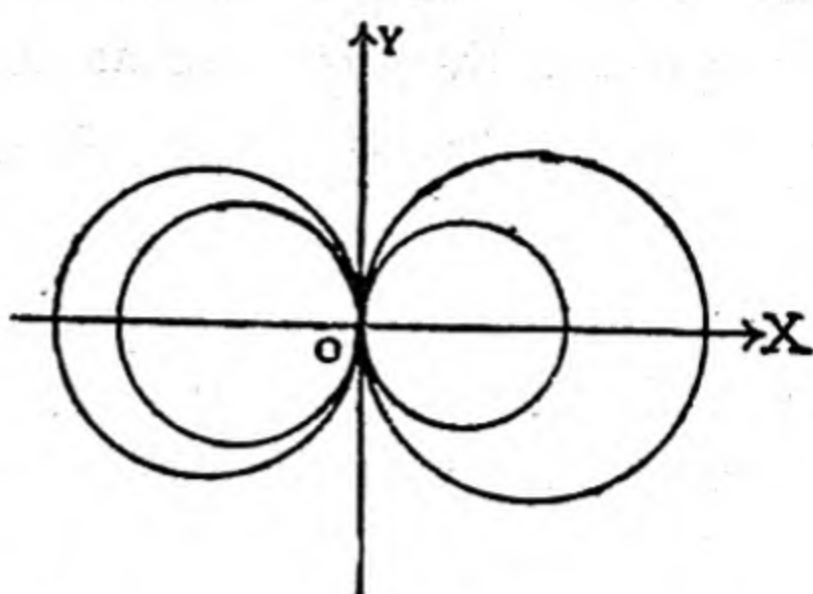


Fig. 5

In this case the coaxal system is said to be of the *common point* (or *intersecting*) kind.

II. If $c > 0$, the circles (1) meet the y -axis in imaginary points, but the system has two real limiting points $L(\sqrt{c}, 0)$ and $L'(-\sqrt{c}, 0)$ [fig. 4]. In this case the coaxal system is said to be of the *limiting point* (or *non-intersecting*) kind.

III. If $c = 0$, the circles (1) touch the y -axis at the origin where the two limiting points also coincide.

Note. The radius of circle (1) i. e., $\sqrt{g^2 - c}$ is imaginary if $g^2 < c$. Hence, no real member of the coaxal system (1) can have its centre between $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$, i. e., between L and L' . This explains the reason for calling the points L and L' as the limiting points of the system (1).

2.4. The Polar of a Limiting Point.

If L and L' be the limiting points of a system of coaxal circles, the polar of either of these points w. r. t. any circle of the system passes through the other. (Patna 1947)

Let the equation of a system of coaxal circles be

$$x^2 + y^2 + 2gx + c = 0, \quad \dots\dots (1)$$

where g is a parameter and c a constant, then the two limiting points of the system (1) are $L(\sqrt{c}, 0)$ and $L'(-\sqrt{c}, 0)$ [§ 2.3]. The polar of L w. r. t. the circle (1) is

$$x\sqrt{c} + y \cdot 0 + g(x + \sqrt{c}) + c = 0,$$

$$\text{or} \quad (\sqrt{c} + g)(x + \sqrt{c}) = 0;$$

$$\therefore \quad x = -\sqrt{c}, \quad [\because g \neq -\sqrt{c}]$$

which represents a straight line through L' parallel to y -axis.

Similarly, it can be shown that the polar of L' w. r. t. (1) passes through L .

It follows that the limiting points of (1) are conjugate points w. r. t. every member of the system.

2.5. Orthogonal Coaxal Systems.

Every circle of a coaxal system is cut orthogonally by every circle passing through the limiting points of the system.

Let the equation of a coaxal system of circles be

$$x^2 + y^2 + 2gx + c = 0, \quad \dots\dots (1)$$

where g is a parameter and c a constant.

Any circle through $(\pm\sqrt{c}, 0)$, the limiting points of (1), has an equation of the form

$$x^2 + y^2 + 2fy - c = 0, \quad \dots\dots (2)$$

since the centre of (2) lies on the y -axis and when $y=0$, $x = \pm\sqrt{c}$.

The circles (1) and (2) are evidently orthogonal to each other, since the condition of § 5.2, Part I,

i. e., $2g \cdot (0) + 2(0) \cdot f = c - c$
is identically satisfied.

The circles (2) also form a coaxal system with f as parameter [§ 2.2 (2)]. Also, the circles (1) pass through $(0, \pm\sqrt{-c})$ which are the limiting points of the system (2).

Hence, each circle of the system (1) is orthogonal to each circle of the system (2), and conversely, *i. e.*, the two systems of coaxal circles are orthogonal. Further, each circle of one system passes through the limiting points (real or imaginary) of the other.

Orthogonal coaxal systems are also called *conjugate systems*.

Note 1. It is evident that if the system (1) has real limiting points, *i. e.*, $c > 0$, then the system (2) has imaginary limiting points and *vice versa*. Thus, of the two orthogonal coaxal systems, if one is of the limiting point kind, the other must be of the common point kind.

Note 2. Since the equations of two circles can be written in the form (§ 5.6, Part I)

$$x^2 + y^2 + 2g_1x + c = 0, \text{ and } x^2 + y^2 + 2g_2x + c = 0,$$

it follows from above that each of these circles is cut orthogonally by every member of the coaxal system (2).

Note 3. As already pointed out in § 5.21 Note 2, Part I, it can be easily shown that the length of the tangent from the

centre of any circle of system (1) to a circle of system (2) is equal to the radius of the first circle and *conversely*.

Ex. 1. Find the coordinates of the limiting points of the coaxal system of which two members are

$$S_1 \equiv x^2 + y^2 + 2x + 4y + 7 = 0 \text{ and } S_2 \equiv x^2 + y^2 + 4x + 2y + 5 = 0.$$

The equation of the coaxal system is $S_2 + \lambda S_1 = 0$,

$$\text{i. e., } x^2 + y^2 + 2 \frac{2+\lambda}{1+\lambda} x + 2 \frac{1+2\lambda}{1+\lambda} y + \frac{5+7\lambda}{1+\lambda} = 0. \quad \dots\dots(1)$$

The radius of this circle is zero, if

$$\left(\frac{2+\lambda}{1+\lambda} \right)^2 + \left(\frac{1+2\lambda}{1+\lambda} \right)^2 - \frac{5+7\lambda}{1+\lambda} = 0, \quad \dots\dots(2)$$

$$\text{or } \lambda^2 + 2\lambda = 0 \quad (\text{on simplification}) ;$$

$$\therefore \lambda = 0, \text{ or } -2.$$

Now, the limiting points of (1) are the centres of its point circles. The centre of (1) is $\left(-\frac{2+\lambda}{1+\lambda}, -\frac{1+2\lambda}{1+\lambda} \right)$. Hence the two limiting points are $(-2, -1)$ and $(0, -3)$, corresponding to the values $\lambda = 0$ and $\lambda = -2$ respectively.

Note. The equation (2) giving the values of λ for limiting points is generally a quadratic. If it reduces to a linear equation, i. e., the coefficient of λ^2 is zero, then one value of λ tends to infinity. Thus, if we take the equation of the coaxal system as $S_1 + \lambda S_2 = 0$ in the above example, it will be found that the condition for limiting points becomes $2\lambda + 1 = 0$;

$\therefore \lambda = -\frac{1}{2}$, or ∞ . Substituting these values of λ in $\left(-\frac{1+2\lambda}{1+\lambda}, -\frac{2+\lambda}{1+\lambda} \right)$, we get the coordinates of the limiting points as $(0, -3)$ and $(-2, -1)$.

Ex. 2. If the circle $x^2 + y^2 - 2\alpha x - 2\beta y + c = 0$ is the circle of a coaxal system having the origin as a limiting point, show that the other limiting point is $\left(\frac{\alpha c}{\alpha^2 + \beta^2}, \frac{\beta c}{\alpha^2 + \beta^2} \right)$, and prove that the equation of this system and the conjugate system are respectively

$$\lambda(x^2 + y^2) - 2\alpha x - 2\beta y + c = 0, \quad (\alpha + \mu\beta)(x^2 + y^2) - c(x + \mu y) = 0, \quad \lambda, \mu \text{ being parameters.}$$

Since origin is a limiting point of the given coaxal system,

the point circle $x^2 + y^2 = 0$ is a member of the system. Hence, the equation of this coaxal system is

$$x^2 + y^2 - 2\alpha x - 2\beta y + c + k(x^2 + y^2) = 0, \quad (k, \text{ a parameter}),$$

$$\text{or} \quad \lambda(x^2 + y^2) - 2\alpha x - 2\beta y + c = 0, \quad \dots\dots (1)$$

where $\lambda = 1 + k$.

$$\text{For limiting points, } \left(\frac{\alpha}{\lambda}\right)^2 + \left(\frac{\beta}{\lambda}\right)^2 - \frac{c}{\lambda} = 0,$$

$$\text{or} \quad \alpha^2 + \beta^2 - c\lambda = 0;$$

$$\therefore \quad \lambda = \infty, \text{ or } \frac{\alpha^2 + \beta^2}{c}$$

The centre of (1) is $\left(\frac{\alpha}{\lambda}, \frac{\beta}{\lambda}\right)$. The origin corresponds to the value $\lambda = \infty$, and the other limiting point is,

$$\left(\frac{\alpha c}{\alpha^2 + \beta^2}, \frac{\beta c}{\alpha^2 + \beta^2}\right).$$

Let the system conjugate to (1) [i. e., orthogonal to (1)] be

$$x^2 + y^2 + 2gx + 2fy + c' = 0, \quad \dots\dots (2)$$

$$\text{then} \quad 2g \cdot \frac{-\alpha}{\lambda} + 2f \cdot \frac{-\beta}{\lambda} = c' + \frac{c}{\lambda},$$

$$\text{or} \quad 2g\alpha + 2f\beta + c + \lambda c' = 0.$$

Since this condition is true for all values of λ , $c' = 0$, and

$$\therefore \quad f = -\frac{c + 2g\alpha}{2\beta}.$$

Substituting these values in (2), the equation of the conjugate system becomes $x^2 + y^2 + 2g \left(x - \frac{c + 2g\alpha}{2g\beta} y\right) = 0$. $\dots\dots (3)$

Writing $\frac{c + 2g\alpha}{2g\beta} = -\mu$, or $2g = \frac{-c}{\alpha + \mu\beta}$, the equation (3) is changed to the form

$$(\alpha + \mu\beta)(x^2 + y^2) - c(x + \mu y) = 0.$$

Exercise 2

1. If two of the lines given by

$$ax^3 + 3bx^2y + 3cxy^2 + dy^3 = 0$$

are at right angles, show that

$$a^2 + 3ac + 3bd + d^2 = 0.$$

(Rajputana '54 ; Allahabad '58)

2. Prove that the equation $m(x^2 - 3y^2)x = y(3x^2 - y^2)$ represents three straight lines equally inclined to each other. (Calcutta 1902)

3. Prove that two of the lines represented by $ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4 = 0$ will bisect the angles between the other two, if $c + 6a = 0$ and $b + d = 0$. (Rajputana 1958)

4. If the lines $ax^2 + 2hxy + by^2 = 0$ be two sides of a parallelogram and the line $lx + my = 1$ be one of its diagonals, show that the equation of the other diagonal is

$$y(bl - hm) = x(am - hl). \quad (\text{Punjab 1955})$$

5. A point moves so that the distance between the feet of the perpendiculars from it on the lines $ax^2 + 2hxy + by^2 = 0$ is a constant $2k$. Show that the equation of its locus is given by $(x^2 + y^2)(h^2 - ab) = k^2[(a - b)^2 + 4h^2]$. (Agra '52 ; Rajputana '57)

6. Find the locus of the orthocentre of a triangle of which two sides are given by the equation $ax^2 + 2hxy + by^2 = 0$ and whose third side passes through a fixed point (f, g) .

7. If (x_1, y_1) be the centroid of a triangle whose sides are the lines $ax^2 + 2hxy + by^2 = 0$ and $lx + my = 1$, show that

$$\frac{x_1}{bl - hm} = \frac{y_1}{am - hl} = \frac{2/3}{bl^2 - 2hlm + am^2}.$$

Hence, find the locus of the centroid of a triangle of which two sides are given by the equation $ax^2 + 2hxy + by^2 = 0$ and whose third side passes through a fixed point.

8. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines, the area of the triangle formed by their bisectors

and the x -axis is $\frac{\sqrt{(a-b)^2 + 4h^2}}{2h} \cdot \frac{ca - g^2}{ab - h^2}$.

(Rajputana '56 ; Agra '58)

9. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines $ax^2 + 2hxy + by^2 = 0$. Prove that the locus of its vertex is $(a + b)(x^2 + y^2) + 2h(fy + gx) + (a - b)(fx - gy) = 0$.

10. A parallelogram is formed by the lines $ax^2 + 2hxy + by^2 = 0$,

and the lines through (p, q) parallel to them. Prove that the equation of the diagonal which does not pass through the origin is

$$(2x - p)(ap + hq) + (2y - q)(hp + bq) = 0.$$

(Agra 1960)

Show also that the area of the parallelogram is

$$(ap^2 + 2hpq + bq^2) / 2\sqrt{h^2 - ab}. \quad (\text{Rajputana 1959})$$

11. If p_1, p_2 be the perpendiculars from (x_1, y_1) on the straight lines $ax^2 + 2hxy + by^2 = 0$, prove that

$$(p_1^2 + p_2^2) [(a - b)^2 + 4h^2] = 2(a - b)(ax_1^2 - by_1^2) + 4h(a + b)x_1y_1 + 4h^2(x_1^2 + y_1^2). \quad (\text{Agra 1959})$$

12. Prove that the system of coaxal circles determined by $S = 0$ and $S' = 0$ can be grouped in pairs of circles orthogonal to one another.

In particular, if a and a' are the radii of $S = 0$ and $S' = 0$, show that $\frac{S}{a} \pm \frac{S'}{a'} = 0$ is one such pair.

(Nagpur '54 Aligarh '59)

13. If A, B, C be the centres of three coaxal circles, and t_1, t_2, t_3 be the tangents to them from any point, prove the relation

$$BC \cdot t_1^2 + CA \cdot t_2^2 + AB \cdot t_3^2 = 0. \quad (\text{Agra '55, '58 S})$$

If r_1, r_2, r_3 be their radii respectively, prove that

$$r_1^2 \cdot BC + r_2^2 \cdot CA + r_3^2 \cdot AB = -AB \cdot BC \cdot CA. \quad (\text{Punjab 1957})$$

14. Show that the points which have the same polar w. r. t. every member of a system of coaxal circles are the limiting points of the system. (Kashmir 1951)

15. Prove that the polars of any point w. r. t. a system of coaxal circles all pass through a fixed point and prove that the two points are equidistant from the radical axis and subtend a right angle at a limiting point of the system. (Nagpur 1947)

16. Show that the chords of intersection of a fixed circle with circles of a given coaxal system pass through a fixed point.

17. Prove that the limiting points of a system of coaxal circles are inverse points w. r. t. every circle of the system.

18. Prove that a common tangent to two circles of a coaxal system subtends a right angle at either limiting point of the system.

19. Prove that the equations of two given circles can always be put in the form

$$x^2 + y^2 + ax + b = 0, \quad x^2 + y^2 + a'x + b = 0,$$

and that one of the circles will be within the other if aa' and b are both positive. (Agra '53 ; Delhi '58)

[Hint : One of the circles lies within the other if the limiting points are real and the two centres lie on the same side of y -axis.]

20. Prove that the limiting points of the system

$$x^2 + y^2 + 2gx + c + \lambda (x^2 + y^2 + 2fy + k) = 0$$

subtend a right angle at the origin, if

$$\frac{c}{g^2} + \frac{k}{f^2} = 2.$$

21. The points $(2, 1)$ and $(1, 2)$ are the limiting points of a system of coaxal circles. Show that no circle of this system can pass through the origin. Is there any exceptional case ?

22. If the origin be one of the limiting points of a system of coaxal circles of which $x^2 + y^2 + 2gx + 2fy + c = 0$ is a member, the equation of the system of circles cutting them all orthogonally is

$$(x^2 + y^2)(g + \mu f) + c(x + \mu y) = 0. \quad (\text{Delhi Hons. '58 ; Rajputana '59})$$

23. Find the limiting points of the system of circles

$$x^2 + y^2 + 2gx + c + \lambda (x^2 + y^2 + 2fy + c') = 0,$$

and show that the square of the distance between them

$$\text{is } \{(c - c')^2 - 4f^2g^2 + 4f^2c + 4g^2c'\} / (f^2 + g^2). \quad (\text{Delhi Hons. 1954})$$

24. Show that as λ varies, the circles

$$x^2 + y^2 + 2ax + 2by + \lambda (ax - by) = 0$$

form a coaxal system. Find the equation of the radical axis.

Find also the equations of the circles which are orthogonal to all the circles of the above system : (Aligarh 1958)

25. Show that the general equation of a circle which touches the two circles

$$x^2 + y^2 + c^2 + 2ax = 0, \quad x^2 + y^2 + c^2 + 2bx = 0$$

may be written in the form

$$\begin{aligned} & \sqrt{[(c^2 + \mu^2)(c^2 + ab)]} (x^2 + y^2 + c^2 + 2\lambda x) \\ & + c\sqrt{ab - \lambda^2} (x^2 + y^2 - c^2 + 2\mu y) = 0, \end{aligned}$$

where μ has any value, and λ is either root of the quadratic equation $(a + b)(c^2 + \lambda^2) = 2\lambda(c^2 + ab)$.

Answers

6. $(a+b)(fx+gy)=bx^2-2hxy+ay^2.$

7. $3(ax^2+2hxy+by^2)=2[(af+hg)x+(hf+bg)y].$

21. Radical axis, which is the limiting case of a member of the system, passes through the origin.

23. $\left(-\frac{g}{1+\lambda}, -\frac{\lambda f}{1+\lambda}\right)$, where λ is a root of

$$(f^2-c')\lambda^2-(c+c')\lambda+g^2-c=0.$$

24. $ax-by=0$; $x^2+y^2+\mu(bx+ay+2ab)=0$, μ a parameter.

CHAPTER III

CHANGE OF AXES

3.1. Changes of Axes. The translation of rectangular axes parallel to themselves is already discussed in Part I (see §§ 1.7, 1.8). We shall now consider below **rotation** of the axes without change of origin.

Note. The change in axes and hence a change in coordinates is also called a *transformation* of coordinates.

3.11. Rotation. Let OX , OY be the given axes and OX' , OY' their new positions obtained by rotating both OX and OY anti-clockwise about O through an angle θ .

Let the coordinates of any point P referred to the old and new axes be respectively (x, y) and (X, Y) .

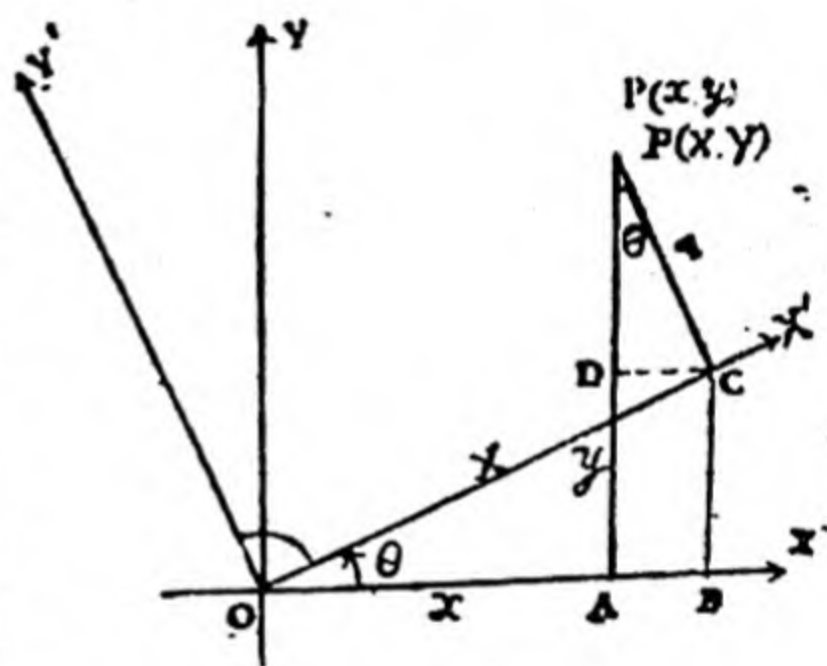


Fig. 6

Draw PA , PC perpendiculars to OX and OX' respectively, and CB , CD perpendiculars to OY and AP respectively (fig. 6).

Then $OA = x$, $AP = y$, $OC = X$, $CP = Y$ and $\angle DPC = \theta$.

From the $\triangle OBC$, we have $OC \cos \theta = OB$,

or $X \cos \theta = OA + AB = x + CD$.

But $CD = Y \sin \theta$, from $\triangle PDC$;

$\therefore X \cos \theta = x + y \sin \theta$,

or $x = X \cos \theta - Y \sin \theta$(1)

Also, $AP = AD + DP = BC + DP$,

or $y = X \sin \theta + Y \cos \theta$,(2)

from $\triangle s OBC$ and PDC .

The equations (1) and (2) are called the equations of transformation.

Note. 1. When translation is followed by rotation of the axes, the transformation can be discussed in two stages, or combining the two, the equations of transformation can be written as

$$\left. \begin{aligned} x &= h + X \cos \theta - Y \sin \theta, \\ y &= k + X \sin \theta + Y \cos \theta, \end{aligned} \right\} \dots\dots(3)$$

where the axes are first translated to parallel positions through (h, k) and then rotated through an angle θ .

Translation and rotation do not in general give the same results when the order of application is interchanged.

Note. 2. Change from one set of rectangular axes to another set of rectangular axes is also called *orthogonal transformation*.

3.12. Removal of the xy -Term from an Equation of the Second Degree. A general equation of the second degree in x and y is $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$. $\dots\dots(1)$

Now, if the origin is changed to (h, k) , the axes remaining parallel to themselves, the equations of transformation are $x = X + h, y = Y + k$.

Substituting these values of x and y in (1), we easily see that the second degree terms are not affected. Since we want to remove the term containing xy , we take the equation of the second degree as

$$ax^2 + 2hxy + by^2 = 0, \dots\dots(2)$$

and keep the origin fixed. Now rotate the axes through an angle θ and let X, Y be the new coordinates (§ 3.11), then

$$\left. \begin{aligned} x &= X \cos \theta - Y \sin \theta, \\ y &= X \sin \theta + Y \cos \theta. \end{aligned} \right\}$$

Substituting these values in (2), we get

$$a(X \cos \theta - Y \sin \theta)^2 + 2h(X \cos \theta - Y \sin \theta)(X \sin \theta + Y \cos \theta) + b(X \sin \theta + Y \cos \theta)^2 = 0,$$

$$\text{or} \quad a'X^2 + 2h'XY + b'Y^2 = 0, \dots\dots(3)$$

$$\left. \begin{aligned} \text{where} \quad a' &= a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta, \\ h' &= h(\cos^2 \theta - \sin^2 \theta) - (a - b) \sin \theta \cos \theta, \\ \text{and} \quad b' &= a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta. \end{aligned} \right\} \dots\dots(4)$$

Hence, the xy -term in (2) can be removed, if

$$h(\cos^2 \theta - \sin^2 \theta) - (a - b) \sin \theta \cos \theta = 0,$$

$$\text{i. e., if} \quad \tan 2\theta = \frac{2h}{a - b}. \dots\dots(5)$$

The equation (5) can always be solved, if a, b, h are real numbers, since $\tan 2\theta$ can have any value between $+\infty$ and $-\infty$.

Hence, the axes can be so changed that $ax^2+2hxy+by^2$ reduces to the form $a'X^2+b'Y^2$.

3.13. Invariants. If by a change of rectangular axes, without change of origin, the expression $ax^2+2hxy+by^2$ is transformed into $a'X^2+2h'XY+b'Y^2$, then

$$a+b=a'+b' \text{ and } ab-h^2=a'b'-h'^2.$$

Let the axes be rotated through an angle θ as in § 3.12, then $ax^2+2hxy+by^2$ is transformed into $a'X^2+2h'XY+b'Y^2$, where

$$\left. \begin{aligned} a' &= a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta, \\ b' &= a \sin^2 \theta - 2h \sin \theta \cos \theta + b \cos^2 \theta, \\ \text{and } h' &= h(\cos^2 \theta - \sin^2 \theta) - (a-b) \sin \theta \cos \theta \\ &= \frac{1}{2} [2h \cos 2\theta - (a-b) \sin 2\theta]. \end{aligned} \right\} \dots\dots(1)$$

From equations (1), we see that

$$\begin{aligned} a'+b' &= a(\cos^2 \theta + \sin^2 \theta) + b(\sin^2 \theta + \cos^2 \theta) \\ &= a+b, \end{aligned} \dots\dots\dots(2)$$

and $a'-b' = (a-b)(\cos^2 \theta - \sin^2 \theta) + 4h \sin \theta \cos \theta$;

$$\begin{aligned} \therefore 4a'b' &= (a'+b')^2 - (a'-b')^2 \\ &= (a+b)^2 - [(a-b) \cos 2\theta + 2h \sin 2\theta]^2 \\ &= (a+b)^2 - (a-b)^2(1 - \sin^2 2\theta) - 4(a-b)h \sin 2\theta \cos 2\theta \\ &\quad - 4h^2(1 - \cos^2 2\theta) \\ &= (a+b)^2 - (a-b)^2 - 4h^2 + [2h \cos 2\theta - (a-b) \sin 2\theta]^2 \\ &= 4ab - 4h^2 + 4h'^2, \end{aligned}$$

$$\text{or } a'b' - h'^2 = ab - h^2. \dots\dots(3)$$

The expressions $(a+b)$ and $(ab-h^2)$ are called *invariants*, because their values remain unchanged when the axes are changed.

Ex. If (x, y) and (X, Y) be the coordinates of the same point referred to two sets of rectangular axes, and if $ux+vy$, when u and v are independent of x, y becomes $u'X+v'Y$, show that

$$u^2+v^2=u'^2+v'^2. \quad (\text{Aligarh 1956})$$

Let one set of rectangular axes be rotated through an angle θ to get the other set, then if (x, y) and (X, Y) are the coordinates of the same point referred to these sets of axes respectively, we have

$$x = X \cos \theta - Y \sin \theta \text{ and } y = X \sin \theta + Y \cos \theta.$$

Since $ux+vy$ is transformed into $u'X+v'Y$, we have

$$u'X + v'Y \equiv u(X \cos \theta - Y \sin \theta) + v(X \sin \theta + Y \cos \theta) \\ \equiv (u \cos \theta + v \sin \theta)X + (-u \sin \theta + v \cos \theta)Y;$$

$$\therefore u' = u \cos \theta + v \sin \theta, \quad \dots \dots (1)$$

$$\text{and } v' = -u \sin \theta + v \cos \theta. \quad \dots \dots (2)$$

$$\text{Hence, } u'^2 + v'^2 = u^2 (\cos^2 \theta + \sin^2 \theta) + v^2 (\sin^2 \theta + \cos^2 \theta) \\ = u^2 + v^2.$$

3.2. Oblique Axes. It is sometimes more convenient to use oblique (or *skewed*) axes in the cartesian coordinate system as in the case of the equation of a hyperbola referred to its asymptotes (see § 6.8). In general, however, the use of oblique axes creates technical complications in the various formulae.

In the articles that follow the change from a set of oblique axes to a rectangular set or another oblique set will be discussed.

3.3. Change from Oblique Axes to Rectangular Axes with the same origin and same x-axis.

Let (x, y) be the coordinates of a point P referred to oblique axes OX, OY inclined at an angle ω and (X, Y) be the coordinates of P when referred to rectangular axes OX, OY' .

If PM, PM' are drawn parallel to OY, OY' respectively to meet the x -axis in M and M' (fig. 7), then

$$OM = OM' - MM' = OM' - M'P \cot \omega,$$

$$\text{and } MP = M'P \operatorname{cosec} \omega,$$

$$\text{i. e., } \left. \begin{aligned} x &= X - Y \cot \omega, \\ y &= Y \operatorname{cosec} \omega. \end{aligned} \right\} \dots \dots (1)$$

Solving these for X and Y , we have

$$\left. \begin{aligned} X &= x + y \cos \omega, \\ Y &= y \sin \omega. \end{aligned} \right\} \dots \dots (2)$$

Note. Similar equations of transformation can be written when the y -axis is common to the two systems of axes.

§ 3.31. Angle between two Straight Lines. To find the angle between the lines $y = m_1x + c_1$ and $y = m_2x + c_2$, the axes being inclined at an angle ω .

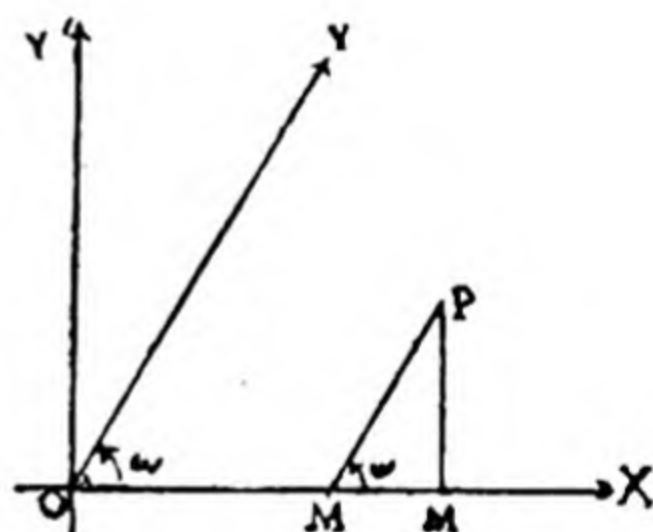


fig. 7

Changing to rectangular axes with the same origin and the same x -axis and denoting the new coordinates by (X, Y) , we have (§ 3.3)

$$x = X - Y \cot \omega \text{ and } y = Y \operatorname{cosec} \omega.$$

The given equations are, therefore, transformed to

$$Y \operatorname{cosec} \omega = m_1 (X - Y \cot \omega) + c_1.$$

and

$$Y \operatorname{cosec} \omega = m_2 (X - Y \cot \omega) + c_2,$$

i. e.,

$$Y = \frac{\sin \omega}{1 + m_1 \cos \omega} (m_1 X + c_1),$$

and

$$Y = \frac{\sin \omega}{1 + m_2 \cos \omega} (m_2 X + c_2),$$

Hence, the angle between the lines

$$\begin{aligned} &= \tan^{-1} \frac{\frac{m_1 \sin \omega}{1 + m_1 \cos \omega} - \frac{m_2 \sin \omega}{1 + m_2 \cos \omega}}{1 + \frac{m_1 m_2 \sin^2 \omega}{(1 + m_1 \cos \omega)(1 + m_2 \cos \omega)}} \\ &= \tan^{-1} \frac{(m_1 - m_2) \sin \omega}{1 + (m_1 + m_2) \cos \omega + m_1 m_2} \quad \dots\dots (1) \end{aligned}$$

Cor. 1. The two given lines are perpendicular to each other, if $1 + (m_1 + m_2) \cos \omega + m_1 m_2 = 0$, $\dots\dots (2)$ which reduces to the already known condition $m_1 m_2 = -1$ when $\omega = 90^\circ$.

Cor. 2. The two given lines are parallel, if $m_1 = m_2$ as in the case of rectangular axes.

Note. When the equations of the given lines are

$$a_1 x + b_1 y + c_1 = 0,$$

and

$$a_2 x + b_2 y + c_2 = 0,$$

the condition of perpendicularity (2) above becomes

$$1 + \left(-\frac{a_1}{b_1} - \frac{a_2}{b_2} \right) \cos \omega + \left(-\frac{a_1}{b_1} \right) \left(-\frac{a_2}{b_2} \right) = 0,$$

$$\text{i. e., } a_1 a_2 + b_1 b_2 - (a_1 b_2 + a_2 b_1) \cos \omega = 0. \quad \dots\dots (3)$$

Writing (3) as $\frac{a_2}{b_1 - a_1 \cos \omega} = -\frac{b_2}{a_1 - b_1 \cos \omega}$, we see

that a straight line perpendicular to $a_1 x + b_1 y + c_1 = 0$ is $(b_1 - a_1 \cos \omega) x - (a_1 - b_1 \cos \omega) y + \lambda = 0$, where λ is a parameter.

§ 3.32. Length of Perpendicular. To find the length of the perpendicular from a given point $P(x_1, y_1)$ on the straight line $ax + by + c = 0$, the axes being inclined at an angle ω .

Changing to rectangular axes with the same origin and the same x -axis and denoting the new coordinates by capital letters, we have (§ 3.3)

$$x = X - Y \cot \omega, \quad y = Y \operatorname{cosec} \omega;$$

and $x_1 = X_1 - Y_1 \cot \omega, \quad y_1 = Y_1 \operatorname{cosec} \omega.$

The transformed equation of the given line is thus

$$a(X - Y \cot \omega) + bY \operatorname{cosec} \omega + c = 0.$$

Hence, the length of the perpendicular from P on the given line

$$\begin{aligned} &= \frac{a(X_1 - Y_1 \cot \omega) + bY_1 \operatorname{cosec} \omega + c}{\sqrt{[a^2 + (b \operatorname{cosec} \omega - a \cot \omega)^2]}} \quad [\text{see § 2.7, Part I}] \\ &= \frac{(ax_1 + by_1 + c) \sin \omega}{\sqrt{[a^2 \sin^2 \omega + (b - a \cos \omega)^2]}} \\ &= \frac{(ax_1 + by_1 + c) \sin \omega}{\sqrt{[a^2 - 2ab \cos \omega + b^2]}} \quad \dots\dots (1) \end{aligned}$$

§ 3.4. Change of Axes in General. It is obvious that the equations of transformation for a mere translation of oblique axes parallel to themselves are the same as for rectangular axes. Thus we need only consider the case *when one set of oblique axes is transformed to another with the same origin.*

Let OX, OY be the old axes containing an angle ω and OX', OY' be the new axes containing an angle ω' and let $X\hat{O}X' = \alpha, X\hat{O}Y' = \beta$, so that $\beta - \alpha = \omega'$. Let the coordinates of a point P be (x, y) with reference to OX, OY and (X, Y) with reference to OX', OY' .

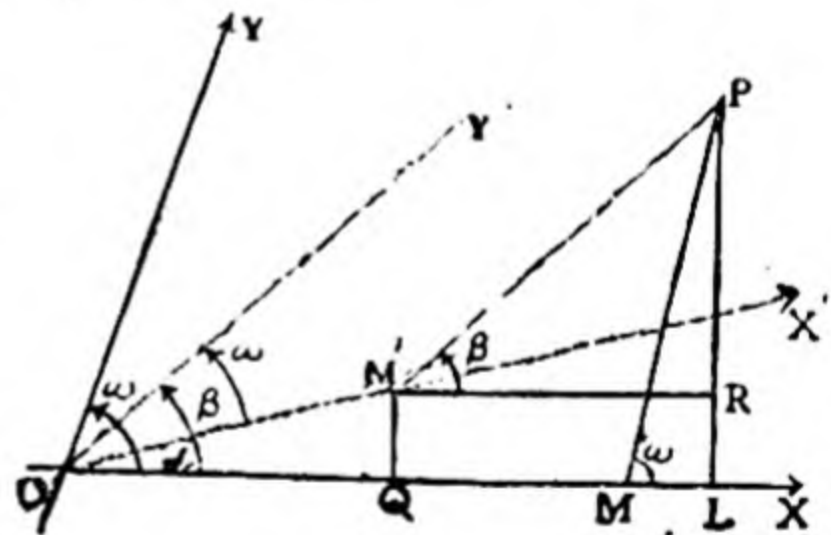


Fig. 8

Draw PM parallel to OY and PM' parallel to OY' to meet OX, OX' in M, M' respectively (fig. 8); then $OM = x$, $MP = y$, and $OM' = X$, $M'P = Y$. Also, draw PL and $M'Q$ perpendicular to OX and $M'R$ perpendicular to PL ; then

$$\begin{aligned} y \sin \omega &= LP = LR + RP = QM' + RP \\ &= X \sin \alpha + Y \sin \beta. \end{aligned}$$

Similarly, since $X'\hat{O}Y = \omega - \alpha$ and $Y'\hat{O}Y = \omega - \beta$, we obtain as above,

$$x \sin \omega = X \sin (\omega - \alpha) + Y \sin (\omega - \beta).$$

Thus, the equations of transformation are

$$\left. \begin{aligned} x &= X \frac{\sin(\omega - \alpha)}{\sin \omega} + Y \frac{\sin(\omega - \beta)}{\sin \omega}, \\ \text{and } y &= X \frac{\sin \alpha}{\sin \omega} + Y \frac{\sin \beta}{\sin \omega} \end{aligned} \right\} \dots\dots(1)$$

Note 1. It is not easy to remember the formulae (1) Moreover these are too complicated to be of any practical use.

It should, however, be observed that the equations of transformation for a change of axes in general with a fixed origin are of the form

$$x = lX + mY,$$

and

$$y = l'X + m'Y;$$

where l, m, l', m' are constants depending upon the directions of the old and new axes.

Note 2. If one of the angles ω, ω' be equal to 90° , we get the transformation of § 3.3 and if $\omega = \omega' = 90^\circ$, we get the transformation of § 3.11.

3.41. The degree of an equation is not altered by any change of axes.

We have seen above that in a change of the axes in general, *i. e.*, when translation and rotation of the axes are combined, the equations of transformation are of the form

$$x = lX + mY + n,$$

and

$$y = l'X + m'Y + n'$$

Since the relations between x and y are linear, it follows that the degree of an equation is not altered by any transformation of the axes.

§ 3.5. Invariants (Oblique Axes). If by any change of axes, without change of origin, the expression $ax^2 + 2hxy + by^2$ is transformed into $a'X^2 + 2h'XY + b'Y^2$, then

$$\frac{a + b - 2h \cos \omega}{\sin^2 \omega} = \frac{a' + b' - 2h' \cos \omega'}{\sin^2 \omega'},$$

and

$$\frac{ab - h^2}{\sin^2 \omega} = \frac{a'b' - h'^2}{\sin^2 \omega'},$$

where ω, ω' are the angles between the old and new axes respectively and $(x, y), (X, Y)$ are the coordinates of the same point P referred to the old and new axes respectively.

Let O be the origin, then the distance OP remains the same in the two systems of axes ;

$$\therefore x^2 + y^2 + 2xy \cos \omega \equiv X^2 + Y^2 + 2XY \cos \omega'. \quad \dots\dots (1)$$

Also, by supposition,

$$ax^2 + 2hxy + by^2 \equiv a'X^2 + 2h'XY + b'Y^2 \quad \dots\dots (2)$$

From (1) and (2), we get

$$\begin{aligned} ax^2 + 2hxy + by^2 + \lambda (x^2 + y^2 + 2xy \cos \omega) \\ \equiv a'X^2 + 2h'XY + b'Y^2 + \lambda (X^2 + Y^2 + 2XY \cos \omega'), \dots\dots (3) \end{aligned}$$

where λ is a parameter.

Since the origin is fixed, the equations of transformation are of the form (§ 3.4)

$$x = lX + mY \text{ and } y = l'X + m'Y.$$

Hence, if any value of λ makes the left-hand side of (3) a perfect square in x and y , viz. $(Ax + By)^2$, then since

$$\begin{aligned} (Ax + By)^2 &= [A(lX + mY) + B(l'X + m'Y)]^2 \\ &= (A'X + B'Y)^2, \text{ say,} \end{aligned}$$

it follows that the same value of λ also makes the right-hand side of (3) a perfect square.

Now, the two sides of (3) are perfect squares, if λ satisfies respectively the equations.

$$(a + \lambda)(b + \lambda) - (h + \lambda \cos \omega)^2 = 0, \quad \dots\dots (4)$$

$$\text{and } (a' + \lambda)(b' + \lambda) - (h' + \lambda \cos \omega')^2 = 0; \quad \dots\dots (5)$$

hence, these quadratic equations in λ must be identical.

Writing the equations (4) and (5) as

$$\lambda^2 \sin^2 \omega + (a + b - 2h \cos \omega) \lambda + ab - h^2 = 0,$$

$$\text{and } \lambda^2 \sin^2 \omega' + (a' + b' - 2h' \cos \omega') \lambda + a'b' - h'^2 = 0$$

and comparing coefficients, we have

$$\frac{a + b - 2h \cos \omega}{\sin^2 \omega} = \frac{a' + b' - 2h' \cos \omega'}{\sin^2 \omega'}, \quad \dots\dots (6)$$

$$\text{and } \frac{ab - h^2}{\sin^2 \omega} = \frac{a'b' - h'^2}{\sin^2 \omega'}. \quad \dots\dots (7)$$

The expressions $\frac{a + b - 2h \cos \omega}{\sin^2 \omega}$ and $\frac{ab - h^2}{\sin^2 \omega}$ are called **invariants**, since their values are not altered by any change of axes.

Note. When $\omega = \omega' = 90^\circ$, the above results reduce to those of § 3.13.

§ 3.51. To find the angle between the pair of lines represented by $ax^2 + 2hxy + by^2 = 0$, the axes being inclined at an angle ω .

Changing to rectangular axes with the same origin, let the expression $ax^2 + 2hxy + by^2$ be transformed to

$a'X^2 + 2h'XY + b'Y^2$, then (§ 3.5)

$$\left. \begin{aligned} \frac{a+b-2h \cos \omega}{\sin^2 \omega} &= \frac{a'+b'-2h' \cos \pi/2}{\sin^2 \pi/2} = a'+b', \\ \text{and } \frac{ab-h^2}{\sin^2 \omega} &= \frac{a'b'-h'^2}{\sin^2 \pi/2} = a'b'-h'^2. \end{aligned} \right\} \dots\dots(1)$$

Now, if θ be the angle between the pair of lines

$$a'X^2 + 2h'XY + b'Y^2 = 0,$$

$$\begin{aligned} \text{then } \tan \theta &= \frac{2\sqrt{h'^2 - a'b'}}{a' + b'} \\ &= \frac{2\sqrt{h^2 - ab} \sin \omega}{a + b - 2h \cos \omega}, \text{ from (1)} \end{aligned}$$

Cor. The given lines will be at right angles, if

$$a + b - 2h \cos \omega = 0.$$

Ex. If the formulae for transformation from one set of axes to another with the same origin be

$$x = lX + mY \text{ and } y = l'X + m'Y,$$

show that

$$\frac{l^2 + l'^2 - 1}{ll'} = \frac{m^2 + m'^2 - 1}{mm'} \quad (\text{Gujrat 1957})$$

Let ω, ω' be the angles between the old and new axes respectively and let $(x, y), (X, Y)$ be the coordinates of the same point P referred to the two sets of axes respectively.

Since the origin O is fixed, the distance OP is the same in the two systems of axes;

$$\therefore x^2 + y^2 + 2xy \cos \omega \equiv X^2 + Y^2 + 2XY \cos \omega'$$

Now, $x = lX + mY$ and $y = l'X + m'Y$;

$$\begin{aligned} \therefore (lX + mY)^2 + (l'X + m'Y)^2 + 2(lX + mY)(l'X + m'Y) \cos \omega \\ \equiv X^2 + Y^2 + 2XY \cos \omega' \end{aligned}$$

Equating the coefficients of X^2 and Y^2 , we have

$$l^2 + l'^2 + 2ll' \cos \omega = 1, \quad \text{or } (l^2 + l'^2 - 1)/ll' = -2 \cos \omega,$$

and $m^2 + m'^2 + 2mm' \cos \omega = 1$, or $(m^2 + m'^2 - 1)/mm' = -2 \cos \omega$.

$$\text{Hence, } \frac{l^2 + l'^2 - 1}{ll'} = \frac{m^2 + m'^2 - 1}{mm'}$$

Exercise 3

1. The equation $3x^2 + 2xy + 3y^2 - 18x - 22y + 50 = 0$ is transformed to $4x^2 + 2y^2 = 1$ when referred to rectangular axes through the point $(2, 3)$. Find the inclination of the latter axes to the former. (Calcutta 1945)

2. Show that if $ax^2 + 2hxy + by^2 = 1$ and $a'x^2 + 2h'xy + b'y^2 = 1$ represent the same conic, then

$$(a-b)^2 + 4h^2 = (a'-b')^2 + 4h'^2. \quad (\text{Aligarh 1956})$$

3. Prove that the transformation of rectangular axes which converts $\frac{X^2}{p} + \frac{Y^2}{q}$ into $ax^2 + 2hxy + by^2$ will convert

$$\frac{X^2}{p-\lambda} + \frac{Y^2}{q-\lambda} \text{ into } \frac{ax^2 + 2hxy + by^2 - \lambda(ab-h^2)(x^2+y^2)}{1-(a+b)\lambda + (ab-h^2)\lambda^2}.$$

(Lucknow '44 ; Osmania '60)

4. Prove that the value of $g^2 + f^2$ in the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

remains unaffected by orthogonal transformation without change of origin.

5. Show that the lines given by

$$ax + by + c = 0 \text{ and } (ax + by)^2 - 3(bx - ay)^2 = 0$$

form the sides of an equilateral triangle.

[**Hint.** Take as new axes the perpendicular lines

$$ax + by = 0 \text{ and } bx - ay = 0.]$$

6. Show that the equation $x \cos \alpha + y \sin \alpha = p$, when the axes are turned through an angle α becomes $x = p$. Interpret this fact.

*7. Prove that the equation of the straight line which passes through the point (h, k) and is perpendicular to the x -axis is

$$x + y \cos \omega = h + k \cos \omega,$$

ω being the angle between the axes.

(Aligarh 1952)

*8. From a given point (h, k) perpendiculars are drawn to the axes and their feet are joined ; prove that the length of the perpendicular from (h, k) upon this line is

$$hk \sin^2 \omega / \sqrt{(h^2 + k^2 + 2hk \cos \omega)},$$

ω being the angle between the axes.

(Allahabad 1955)

***9.** If (x, y) and (X, Y) , be the coordinates of the same point referred to two sets of axes with the same origin and $lx + my$ be transformed to $l'X + m'Y$, then

$$\frac{l^2 - 2lm \cos \omega + m^2}{\sin^2 \omega} = \frac{l'^2 - 2l'm' \cos \omega' + m'^2}{\sin^2 \omega'},$$

where ω and ω' are the angles between the axes in the two sets.

$$\begin{aligned} \text{[Hint : } (lx + my)^2 + \lambda (x^2 + y^2 + 2xy \cos \omega) \\ \equiv (l'X + m'Y)^2 + \lambda (X^2 + Y^2 + 2XY \cos \omega'). \end{aligned}$$

The same value of λ makes each side a perfect square

Proceed as in § 3.5.]

***10.** Show that the equation of the bisectors of the angles between the lines $ax^2 + 2hxy + by^2 = 0$

$$\text{is } \begin{vmatrix} ax + hy & hx + by \\ x + y \cos \omega & y + x \cos \omega \end{vmatrix} = 0$$

the axes being inclined at an angle ω .

[Hint : Apply the method of invariants.]

Answers

1. 45°

CHAPTER IV

GENERAL EQUATION OF THE SECOND DEGREE

4.1. Equation of a Conic*. We have seen in § 7.11 Part I that a conic of eccentricity e is represented by the general equation of the second degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots\dots (1)$$

In Ch. VII we shall show that the equation (1) always represents a conic.

We shall now establish certain relations applicable to all conics which can be as easily proved for the equation (1) as for the standard forms of the equations of a parabola, an ellipse and a hyperbola.

4.2. Intersection of a Straight Line and a Conic.

Let the equation of the conic be

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots\dots (1)$$

and the equation of a straight line through a given point $P(x_1, y_1)$ be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r, \quad \dots\dots (2)$$

where l, m are the direction cosines of the line and r is the distance of (x, y) from P .

Now, any point on the line (2) is $(x_1 + lr, y_1 + mr)$ which lies on the conic (1), if

$$a(x_1 + lr)^2 + 2h(x_1 + lr)(y_1 + mr) + b(y_1 + mr)^2 + 2g(x_1 + lr) + 2f(y_1 + mr) + c = 0,$$

$$\text{or} \quad r^2(al^2 + 2hlm + bm^2) + 2r[(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m] + ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c = 0, \quad \dots\dots (3)$$

which is a quadratic in r and gives two values of r , real or imaginary.

Hence, the line (2) meets the conic (1) in two points, real or imaginary.

*For definition of a conic, see § 6.1, Part I.

By Calculus,

$$F(x_1 + lr, y_1 + mr) = F(x_1, y_1) + r \left(l \frac{\partial}{\partial x_1} + m \frac{\partial}{\partial y_1} \right) F \\ + \frac{r^2}{2} \left(l \frac{\partial}{\partial x_1} + m \frac{\partial}{\partial y_1} \right)^2 F.$$

Hence, if $S \equiv F(x, y)$, the equation (3) above can be easily obtained.

4.3. Tangent at a Given Point $P(x_1, y_1)$ on the Conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \dots\dots (1)$$

From § 4.2, we see that the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \dots\dots (2)$$

meets the conic (1) in points whose distances from P are the roots of the quadratic.

$$r^2 (al^2 + 2hlm + bm^2) + 2r [(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m] + S_1 = 0, \dots\dots (3)$$

where

$$S_1 \equiv ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c,$$

Since P lies on the conic, $S_1 = 0$ and, therefore, one root of (2) is zero. The line (2) meets the conic (1) in two coincident points, if the second root of (3) is also zero; hence

$$(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0 \dots\dots (4)$$

is the condition that the line (2) should be a tangent to (1) at P .

Eliminating l, m from (2) and (4), we get

$$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0, \dots\dots (5)$$

or

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c \\ = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \\ = 0, \dots\dots (6)$$

which is the standard form of the equation of the tangent to (1) at P .

For the sake of convenience, the equation (6) is briefly written as $T = 0$,

where

$$T \equiv axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c.$$

In calculus notation, the equation (5) is written as

$$(x - x_1) \frac{\partial F}{\partial x_1} + (y - y_1) \frac{\partial F}{\partial y_1} = 0,$$

where

$$F(x_1, y_1) \equiv S.$$

Note. The equation of the tangent to a conic at (x_1, y_1) can be easily written by remembering the following rule :

In the equation of the conic, change x^2 to xx_1 , y^2 to yy_1 , $2xy$ to $(xy_1 + x_1y)$, $2x$ to $x + x_1$ and $2y$ to $y + y_1$, and retain the various coefficients and the constant term.

Aliter. The equation of the chord joining the points (x_1, y_1) and (x_2, y_2) on the conic (1) can be written as

$$\begin{aligned} & a(x-x_1)(x-x_2) + h[(x-x_1)(y-y_2) + (x-x_2)(y-y_1)] + b(y-y_1)(y-y_2) \\ & = ax^2 + 2hxy + by^2 + 2gx + 2fy + c, \end{aligned}$$

since, on simplification it reduces to a linear equation in x and y and on putting $x=x_1, y=y_1$, or $x=x_2, y=y_2$, the two sides vanish.

Now, putting $x_2=x_1$ and $y_2=y_1$, we have the equation of the tangent at (x_1, y_1) as

$$\begin{aligned} & a(x-x_1)^2 + 2h(x-x_1)(y-y_1) + b(y-y_1)^2 \\ & = ax^2 + 2hxy + by^2 + 2gx + 2fy + c \end{aligned}$$

or

$$\begin{aligned} & 2[axx_1 + h(xy_1 + x_1y) + byy_1 + gy + fy] + c \\ & = ax_1^2 + 2hx_1y_1 + by_1^2 \\ & = -2gx_1 - 2fy_1 - c, \quad [\because (x_1, y_1) \text{ lies on (1)}] \end{aligned}$$

i. e.,

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x+x_1) + f(y+y_1) + c = 0.$$

4.31. Condition of Tangency. To find the condition that the line $lx + my + n = 0$ (1)

may touch the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ (2)

If (1) touches (2) at (x_1, y_1) , then it must be identical with

$$\begin{aligned} T \equiv & axx_1 + h(xy_1 + x_1y) + byy_1 + g(x+x_1) + f(y+y_1) \\ & + c = 0, \end{aligned}$$

$$i. e., (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0. \quad \dots\dots (3)$$

Comparing coefficients in (1) and (3), we have

$$\frac{ax_1 + hy_1 + g}{l} = \frac{hx_1 + by_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n} = \lambda, \text{ say ;}$$

then

$$\left. \begin{aligned} ax_1 + hy_1 + g - \lambda l &= 0, \\ hx_1 + by_1 + f - \lambda m &= 0, \\ gx_1 + fy_1 + c - \lambda n &= 0. \end{aligned} \right\} \dots\dots (4)$$

Also,
because

$$\begin{aligned} lx_1 + my_1 + n &= 0, \\ (x_1, y_1) \text{ lies on (1)} & \end{aligned} \quad \dots\dots (5)$$

Eliminating x_1, y_1, λ from (4) and (5), we get

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0, \quad \dots \dots (6)$$

which is the required condition.

Expanding the determinant, we can write this condition as

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$

where A, B, C , etc. are the co-factors of a, b, c , etc. in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Note. We can find the coordinates of the point of contact by showing any two of the equations.

$$\frac{ax_1 + hy_1 + g}{l} = \frac{hx_1 + by_1 + f}{m} = \frac{gx_1 + fy_1 + c}{n}$$

4.4. Tangents from a Point not on the Conic.

Method I. From § 4.2, we see that the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r \quad \dots \dots (1)$$

meets the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots \dots (2)$

in points whose distances from (x_1, y_1) are the roots of $r^2(al^2 + 2hlm + bm^2) + 2r[(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m] + S_1 = 0$.

If (x_1, y_1) does not lie on the conic and the line (1) is a tangent to the conic (2) then the above quadratic in r must have equal roots ;

$$\therefore (al^2 + 2hlm + bm^2)S_1 = [(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m]^2 \quad \dots (3)$$

Eliminating l, m from (1) and (3), we get

$$\begin{aligned} & [a(x - x_1)^2 + 2h(x - x_1)(y - y_1) + b(y - y_1)^2] S_1 \\ & = [(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1)]^2, \end{aligned}$$

$$\text{or} \quad (S + S_1 - 2T) S_1 = (T - S_1)^2,$$

$$\text{or} \quad SS_1 = T^2 \quad \dots (4)$$

which is an equation of the second degree in x and y .

Since the equation (4) represents the locus of points lying on a tangent to the conic from the point (x_1, y_1) , it follows that it is the joint equation of the pair of tangents from (x_1, y_1) to the conic. It also follows that two tangents can be drawn to

a conic from a point not lying on it. These tangents will be real or imaginary depending upon the position of the point w. r. t. the conic.

Method II. Let A be the point of contact of a tangent from $P(x_1, y_1)$ and let $Q(x, y)$ be any point on the tangent PA . If $PA : AQ = \lambda : 1$, then the coordinates of A are

$$\left(\frac{\lambda x + x_1}{\lambda + 1}, \frac{\lambda y + y_1}{\lambda + 1} \right).$$

Since A lies on the conic (2), we have

$$a \left(\frac{\lambda x + x_1}{\lambda + 1} \right)^2 + 2h \left(\frac{\lambda x + x_1}{\lambda + 1} \right) \left(\frac{\lambda y + y_1}{\lambda + 1} \right) + b \left(\frac{\lambda y + y_1}{\lambda + 1} \right)^2 \\ + 2g \left(\frac{\lambda x + x_1}{\lambda + 1} \right) + 2f \left(\frac{\lambda y + y_1}{\lambda + 1} \right) + c = 0,$$

$$\text{or } a(\lambda x + x_1)^2 + 2h(\lambda x + x_1)(\lambda y + y_1) + b(\lambda y + y_1)^2 + 2g(\lambda x + x_1) \\ + 2f(\lambda y + y_1) + c(\lambda + 1)^2 = 0, \\ \text{or } \lambda^2 S + 2\lambda T + S_1 = 0, \dots\dots (5)$$

The roots of this quadratic in λ give the ratios in which PQ is divided at its points of intersection with the conic. Since PQ is a tangent, the equation (5) has equal roots ;

$$\therefore SS_1 = T^2,$$

which being the locus of Q is the joint equation of the pair of tangents from P to the conic (2).

4.41. Locus of Point of Intersection of Perpendicular Tangents.

The equation of the pair of tangents from $P(x_1, y_1)$ to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \dots\dots (1)$$

$$\text{is } (ax^2 + 2hxy + by^2 + 2gx + 2fy + c)(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c)$$

$$= [axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c]^2, \dots\dots (2)$$

by § 4.4.

These tangents are perpendicular to each other, if the sum of the coefficients of x^2 and y^2 in (2) is zero,

$$\text{i. e., if } (a + b)(ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c) \\ = (ax_1^2 + hy_1 + g)^2 + (hx_1 + by_1 + f)^2,$$

$$\text{or } (ab - h^2)(x_1^2 + y_1^2) + 2(bg - hf)x_1 + 2(af - gh)y_1 \\ + c(a + b) - f^2 - g^2 = 0,$$

whence the required locus is the circle

$$(ab-h^2)(x^2+y^2)+2(bg-hf)x+2(af-gh)y+c(a+b)-f^2-g^2=0, \quad \dots\dots (3)$$

The circle (3) is called the *director circle* of the conic (1).

Note 1. The equation (3) can be written as

$$C(x^2+y^2)-2Gx-2Fy+A+B=0, \quad \dots\dots (4)$$

where A, B, C , etc. are the co-factors of a, b, c , etc. in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Note 2. The centre of the circle (3) is

$$\left(\frac{hf-bg}{ab-h^2}, \frac{gh-af}{ab-h^2} \right)$$

which is also the centre of the conic (1), as will be shown in § 4.7. Hence, the *director circle* is concentric with the conic.

4.5. Chord of Contact. To find the equation of the chord of contact of tangents drawn from $P(x_1, y_1)$ to the conic

$$ax^2+2hxy+by^2+2gx+2fy+c=0. \quad \dots\dots (1)$$

Let $A(x', y')$ and $B(x'', y'')$ be the points of contact of tangents drawn from P to the conic (1). Now, the equation of the tangent at A is

$$axx'+h(xy'+x'y)+byy'+g(x+x')+f(y+y')+c=0.$$

Since this passes through P , we have

$$ax_1x'+h(x_1y'+x'y_1)+by_1y'+g(x_1+x')+f(y_1+y')+c=0. \quad \dots\dots (1)$$

Similarly,

$$ax_2x'+h(x_2y'+x'y_2)+by_2y'+g(x_2+x')+f(y_2+y')+c=0. \quad \dots\dots (2)$$

The equations (1) and (2) are also the conditions that the points A and B should be on the straight line

$$axx_1+h(xy_1+x_1y)+byy_1+g(x+x_1)+f(y+y_1)+c=0, \quad \dots\dots (3)$$

which, therefore, is the required equation.

Note. The points of contact of tangents from P can be real or imaginary depending upon the position of P w. r. t. the conic. However, the chord of contact is always a real straight line. Also see Note, § 4.3, Part I.

4.6. Pole and Polar. For definitions, see § 4.4, Part I.

To find the equation of the polar of $P (x_1, y_1)$ w. r. t. the conic.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

Let any straight line through P meet the conic in the points A and B , and let the tangents at A and B (the extremities of the chord AB) meet in $Q (x', y')$, then the straight line AB is the chord of contact of tangents from an external point Q . The equation of AB , therefore, is

$$axx' + h(xy' + x'y) + byy' + g(x + x') + f(y + y') + c = 0.$$

But AB passes through P ;

$$\therefore ax_1x' + h(x_1y' + x'y_1) + by_1y' + g(x_1 + x') + f(y_1 + y') + c = 0.$$

Hence, the locus of $Q (x', y')$, i. e., the polar of P w. r. t. the conic is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0. \dots\dots(1)$$

The equation (1) being linear in x and y represents a straight line.

Note. See Notes 1 and 2, § 4.41, Part I.

4.61. Conjugate Points and Conjugate Lines. We shall first show that if the polar of a point P w. r. t. a given conic passes through Q , then the polar of Q w. r. t. the same conic passes through P .

Let P and Q be the points (x_1, y_1) and (x_2, y_2) respectively and let the equation of the given conic be

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \dots\dots (1)$$

then the polar of P w. r. t. (1) is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

If this passes through Q , then

$$ax_1x_2 + h(x_1y_2 + x_2y_1) + by_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0, \dots\dots (2)$$

which is the condition that P lies on the straight line

$$axx_2 + h(xy_2 + x_2y) + byy_2 + g(x + x_2) + f(y + y_2) + c = 0. \dots\dots (3)$$

But (3) is the polar of Q w. r. t. (1).

Hence, the polar of Q passes through P . The points P and Q are said to be *conjugate points* w. r. t. the conic (1).

The result above can also be stated as :

If the pole of a straight line AB w. r. t. a given conic lies on the

straight line CD , then the pole of CD w. r. t. the same conic lies on AB .

Two such lines are said to be *conjugate lines*.

The proof is easy and is left to the reader.

Note. A *self-conjugate* (or self-polar) triangle is defined as in § 4.45, Part I.

4.62. To find the condition that the two lines

$$l_1x + m_1y + n_1 = 0, \quad \dots\dots (1)$$

$$\text{and} \quad l_2x + m_2y + n_2 = 0, \quad \dots\dots (2)$$

should be conjugate w. r. t. the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots\dots (3)$$

Let (x_1, y_1) be the pole of (1) w. r. t. (2), then (1) is identical with

$$\begin{aligned} & axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0, \\ \text{or} \quad & (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + gx_1 + fy_1 + c = 0. \quad \dots\dots (4) \end{aligned}$$

Comparing coefficients in (1) and (4), we have

$$\frac{ax_1 + hy_1 + g}{l_1} = \frac{hx_1 + by_1 + f}{m_1} = \frac{gx_1 + fy_1 + c}{n_1} = \lambda, \text{ say ;}$$

$$\begin{aligned} \text{then} \quad & \left. \begin{aligned} ax_1 + hy_1 + g - l_1\lambda &= 0, \\ hx_1 + by_1 + f - m_1\lambda &= 0, \\ gx_1 + fy_1 + c - n_1\lambda &= 0. \end{aligned} \right\} \dots\dots (5) \end{aligned}$$

$$\text{and} \quad \text{Also} \quad l_2x_1 + m_2y_1 + n_2 = 0, \quad \dots\dots (6)$$

since the pole of (1) lies on (2), the two lines being conjugate lines.

Eliminating x_1, y_1 and λ from (5) and (6), we get

$$\begin{vmatrix} a & h & g & l_1 \\ h & b & f & m_1 \\ g & f & c & n_1 \\ l_2 & m_2 & n_2 & 0 \end{vmatrix} = 0, \quad \dots\dots (7)$$

which is the required condition.

4.7. Centre of a conic. Let $C(x, y)$ be the centre of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ $\dots\dots (1)$

and let a chord through C be

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = r, \quad \dots\dots (2)$$

The line (2) meets the conic (1) in points whose distances from C are the roots of the equation (see § 4.2)

$$r^2 (al^2 + 2hlm + bm^2) + 2r [(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m] + S_1 = 0. \dots\dots (3)$$

Since C is the centre, every chord through C meets the conic in two points equidistant from C i. e., the roots of (3) are equal in magnitude but opposite in sign ;

$$\therefore (ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0$$

for all values of l and m .

$$\text{Hence,} \quad ax_1 + hy_1 + g = 0, \quad \dots\dots (4)$$

$$\text{and} \quad hx_1 + by_1 + f = 0. \quad \dots\dots (5)$$

Solving (4) and (5), we have

$$x_1 = \frac{hf - bg}{ab - h^2}, \quad y_1 = \frac{gh - af}{ab - h^2}.$$

The coordinates of the centre can also be written as

$$\left(\frac{G}{C}, \frac{F}{C} \right),$$

where G, F, C are respectively the co-factors of g, f, c in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Note. In calculus notation the equations (4) and (5) above can be written as $\frac{\partial F}{\partial x_1} = 0, \frac{\partial F}{\partial y_1} = 0,$

where $F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c.$

4.71. Equation of a Conic referred to its Centre.

The centre (x_1, y_1) of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots\dots (1)$$

is given by the equations (§ 4.7)

$$ax_1 + hy_1 + g = 0, \quad \dots\dots (2)$$

$$\text{and} \quad hx_1 + by_1 + f = 0. \quad \dots\dots (3)$$

If now we transfer the origin to (x_1, y_1) keeping the axes parallel to their original directions, then equation (1) becomes

$$a(X + x_1)^2 + 2h(X + x_1)(Y + y_1) + b(Y + y_1)^2 + 2g(X + x_1) + 2f(Y + y_1) + c = 0,$$

$$\text{or} \quad aX^2 + 2hXY + bY^2 + S_1 = 0 \quad [\because \text{ of (2) and (3) }].$$

$$\begin{aligned} \text{Now,} \quad S_1 &= ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \\ &= (ax_1 + hy_1 + g)x_1 + (hx_1 + by_1 + f)y_1 + gx_1 + fy_1 + c \\ &= gx_1 + fy_1 + c, \end{aligned}$$

or $gx_1 + fy_1 + c - S_1 = 0. \quad \dots\dots(4)$

Eliminating x_1, y_1 from (2), (3) and (4), we get

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c - S_1 \end{vmatrix} = 0,$$

or $\Delta + \begin{vmatrix} a & h & 0 \\ h & b & 0 \\ g & f & -S_1 \end{vmatrix} = 0$, where $\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$;

$$\therefore S_1 = \frac{\Delta}{ab - h^2}.$$

Hence, the equation (1), when referred to parallel axes through the centre, becomes

$$aX^2 + 2hXY + bY^2 + \frac{\Delta}{ab - h^2} = 0, \quad \dots\dots(5)$$

which is of the form

$$Ax^2 + 2Hxy + By^2 = 1. \quad \dots\dots(6)$$

Note. 1. When the coordinates of the centre are known, the value $gx_1 + fy_1 + c$ of S_1 is used and not $\frac{\Delta}{ab - h^2}$.

Note. 2. The equation (6) represents a conic whose centre is at the origin.

4.8. Equation of a Chord in Terms of its Middle Point.

Let $M(x_1, y_1)$ be the middle point of a chord AB of the conic $S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, $\dots\dots(1)$

Any straight line through M is $\frac{x - x_1}{l} = \frac{y - y_1}{m} = r. \quad \dots\dots(2)$

If the line (2) is the chord AB , then the distances MA, MB are the roots of the equation (see § 4.2)

$$r^2(al^2 + 2hlm + bm^2) + 2r[(ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m] + S_1 = 0 \quad \dots\dots(3)$$

Since the segments MA and MB are equal in magnitude but in opposite directions, the sum of the roots of (3) is zero ;

$$\therefore (ax_1 + hy_1 + g)l + (hx_1 + by_1 + f)m = 0. \quad \dots\dots(4)$$

Eliminating l, m from (2) and (4), we get the required equation as

$$(ax_1 + hy_1 + g)(x - x_1) + (hx_1 + by_1 + f)(y - y_1) = 0,$$

$$\text{or} \quad axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c \\ = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c, \quad \dots\dots (5)$$

$$\text{or} \quad T = S_1 \quad \dots\dots (6)$$

4.81. Locus of the Mid-Points of a System of Parallel Chords. Let (x_1, y_1) be the mid-point of one of the parallel chords of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \quad \dots\dots (1)$$

then its equation is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c \\ = ax_1^2 + 2hx_1y_1 + by_1^2 + 2gx_1 + 2fy_1 + c \quad (\S 4.7),$$

$$\text{i.e.,} \quad (ax_1 + hy_1 + g)x + (hx_1 + by_1 + f)y + c = S_1.$$

Since the chords, being all parallel, have the same slope,

$$-\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f} = \text{a constant, say } m.$$

Hence, the locus of (x_1, y_1) is

$$-\frac{ax + hy + g}{hx + by + f} = m,$$

$$\text{i.e.,} \quad (ax + hy + g) + m(hx + by + f) = 0, \quad \dots\dots (2)$$

which is a straight line through the intersection of the lines

$$ax + hy + g = 0 \text{ and } hx + by + f = 0,$$

i. e., a straight line through the centre of the conic as shown in § 4.7.

Note. The locus of the mid-points of a system of parallel chords of a conic is called a *diameter* of the conic. From equation, it follows that every diameter of a conic passes through its centre.

4.82. Conjugate Diameters.

The locus of the mid-points of a system of chords parallel to $y = mx$ is the diameter (§ 4.81)

$$(ax + hy + g) + m(hx + by + f) = 0.$$

If m' be the slope of this diameter, then

$$m' = -\frac{a + hm}{h + bm},$$

$$\text{i. e.,} \quad a + h(m + m') + bmm' = 0. \quad \dots\dots (1)$$

From the symmetry of this result, it follows that the diameter parallel to $y = mx$ bisects chords parallel to $y = m'x$.

Two diameters of a conic each of which bisects chords parallel to the other are called *conjugate diameters*.

Equation (1)' is the condition that diameters parallel to $y=mx$ and $y=m'x$ of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

should be conjugate to each other.

Note. It can be easily seen that the conjugate diameters of a conic are conjugate lines [by using the condition of § 4.62].

Ex. 1. Find the locus of the foot of the perpendicular let fall from the origin upon the tangents to the conic

$$ax^2 + 2hxy + by^2 = 2x.$$

Method I. If the foot of the perpendicular drawn from the origin to a straight line be (x_1, y_1) , then the equation of the

straight line is $y - y_1 = -\frac{x_1}{y_1}(x - x_1)$,

or $xx_1 + yy_1 - (x_1^2 + y_1^2) = 0$.

This line will be a tangent to the given conic, if

$$\begin{vmatrix} a & h & -1 & x_1 \\ h & b & 0 & y_1 \\ -1 & 0 & 0 & -(x_1^2 + y_1^2) \\ x_1 & y_1 & -(x_1^2 + y_1^2) & 0 \end{vmatrix} = 0, \text{ [see § 4.31]}$$

i. e., if $(h^2 - ab)(x_1^2 + y_1^2)^2 + 2(bx_1 - hy_1)(x_1^2 + y_1^2) + y_1^2 = 0$,
on expansion,

Hence, the required locus is

$$(h^2 - ab)(x^2 + y^2)^2 + 2(bx - hy)(x^2 + y^2) + y^2 = 0.$$

Method II. The foot of the perpendicular from the origin upon the straight line $x \cos a + y \sin a = p$ (1)
is $(p \cos a, p \sin a)$.

Eliminating y from (1) and the given conic

$$ax^2 + 2hxy + by^2 - 2x = 0, \text{ (2)}$$

we get $ax^2 + 2hx \left(\frac{p - x \cos a}{\sin a} \right) + b \left(\frac{p - x \cos a}{\sin a} \right)^2 - 2x = 0$,

or $(a \sin^2 a - 2h \sin a \cos a + b \cos^2 a) x^2 + 2(hp \sin a - bp \cos a - \sin^2 a) x + bp^2 = 0$; (3)

\therefore the line (1) touches the conic (2), if (3) has equal roots,

i. e., if $(hp \sin a - bp \cos a - \sin^2 a)^2 = bp^2 (a \sin^2 a - 2h \sin a \cos a + b \cos^2 a)$ (4)

Now, changing $p \cos a$ to x , $p \sin a$ to y and $\sin^2 a$ to $\frac{y^2}{x^2+y^2}$ in (4), the locus of $(p \cos a, p \sin a)$ is

$$\left(hy - bx - \frac{y^2}{x^2+y^2} \right)^2 = b (ay^2 - 2hxy + bx^2),$$

$$\text{or } (hy - bx)^2 + 2y^2 (bx - hy)/(x^2 + y^2) + y^4/(x^2 + y^2)^2 = b (ay^2 - 2hxy + bx^2),$$

$$\text{or } (h^2 - ab) (x^2 + y^2)^2 + 2 (bx - hy) (x^2 + y^2) + y^2 = 0.$$

Ex. 2. If any chord of a conic be drawn through a point O , prove that it will be cut harmonically by the curve and the polar of O .

(Punjab 1938)

Let any chord of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots \dots (1)$$

cut it in P, R and the polar of O w. r. t. it in Q . Let O be the origin and the straight line $OPQR$ the x -axis.

Now, the chord $OPQR$, i. e., the x -axis meets the conic (1), where

$$ax^2 + 2gx + c = 0;$$

$$\therefore OP + OR = -2g/a, \quad OP \cdot OR = c/a,$$

$$\text{and } \frac{1}{OP} + \frac{1}{OR} = -\frac{2g}{a} \cdot \frac{a}{c} = -\frac{2g}{c} \quad \dots \dots (2)$$

Again, the polar of O w. r. t. (1) is

$$gx + fy + c = 0 \quad (\because O \text{ is the origin}).$$

This meets the x -axis, where $gx + c = 0$;

$$\therefore OQ = -c/g. \quad \dots \dots (3)$$

$$\text{Hence, } \frac{1}{OP} + \frac{1}{OR} = \frac{2}{OQ} \quad \dots \dots (4)$$

Note. The relation (4) is sometimes used to define the polar of a point and thence to find its equation.

Thus, if a point Q is taken on the secant OPR of a conic then the locus of Q , subject to the condition (4), is called the polar of Q w. r. t. the conic.

Ex. 3. Pairs of tangents are drawn to the conic $ax^2 + \beta y^2 = 1$ so as to be always parallel to conjugate diameters of the conic

$$ax^2 + 2hxy + by^2 = 1;$$

show that the locus of their point of intersection is the conic

$$ax^2 + 2hxy + by^2 = \frac{a}{\alpha} + \frac{b}{\beta}. \quad (\text{Banaras 1948})$$

The equation of the pair of tangents from (x_1, y_1) to the conic

$$ax^2 + \beta y^2 = 1$$

is

$$(ax^2 + \beta y^2 - 1)(ax_1^2 + \beta y_1^2 - 1) = (axx_1 + \beta yy_1 - 1)^2,$$

or

$$a(\beta y_1^2 - 1)x^2 - 2a\beta x_1 y_1 xy + \beta(ax_1^2 - 1)y^2 + 2ax_1x + 2\beta y_1y - (ax_1^2 + \beta y_1^2) = 0,$$

These tangents are parallel to the lines

$$y = mx \text{ and } y = m'x, \quad \dots\dots (1)$$

$$\text{if } m + m' = 2a\beta x_1 y_1 / [a(\beta y_1^2 - 1)], \quad \dots\dots (2)$$

$$\text{and } mm' = \frac{a(\beta y_1^2 - 1)}{\beta(ax_1^2 - 1)}. \quad \dots\dots (3)$$

Now, the lines (1) are conjugate diameters of the conic

$$ax^2 + 2hxy + by^2 = 1,$$

$$\text{if } a + h(m + m') + bmm' = 0,$$

$$\text{i.e., if } a\beta(ax_1^2 - 1) + 2ha\beta x_1 y_1 + ba(\beta y_1^2 - 1) = 0$$

[from (2) and (3)]

Hence, the required locus is

$$a\beta(ax^2 - 1) + 2ha\beta xy + ba(\beta y^2 - 1) = 0,$$

$$\text{or } a\beta(ax^2 + 2hxy + by^2) = a\beta + ba$$

$$\text{or } ax^2 + 2hxy + by^2 = \frac{a}{\beta} + \frac{b}{a}.$$

Exercise 4

1. Prove that the tangent to the parabola $\sqrt{\frac{x}{h}} + \sqrt{\frac{y}{k}} = 1$ at the point (x_1, y_1) on it is

$$x/\sqrt{(hx_1)} + y/\sqrt{(ky_1)} = 1. \quad (\text{Lucknow 1943})$$

[**Hint** : Take a chord joining the points (x_1, y_1) and (x_2, y_2) on the parabola and find the limiting value of $\frac{y_2 - y_1}{x_2 - x_1}$ as $x_2 \rightarrow x_1$ and $y_2 \rightarrow y_1$.]

2. If the straight line $lx + my = 1$ touches the conic

$$(ax - by)^2 - 2(a^2 + b^2)(ax + by) + (a^2 + b^2)^2 = 0$$

and cuts the axes in A and B , show that AB subtends a right angle at the point (a, b) .

3. Find the equation of the normal at (x_1, y_1) on the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

4. Show that the chord of contact of tangents, drawn from an external point to the conic $ax^2 + 2hxy + by^2 = 1$, subtends a right angle at the centre if the point lies on the conic

$$x^2(a^2 + h^2) + 2h(a + b)xy + y^2(h^2 + b^2) = a + b.$$

5. Prove that the points on the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

the tangents at which are parallel to the line $y = mx$, are the extremities of the diameter.

6. A pair of tangents to the conic $ax^2 + by^2 = 1$ intercept a constant distance $2c$ on the x -axis; prove that the locus of their point of intersection is the curve

$$by^2 (ax^2 + by^2 - 1) = ac^2 (by^2 - 1)^2.$$

7. Find the locus of the poles of the tangents to the conic

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \text{w. r. t. the conic}$$

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

8. Prove that the polar of a given point w. r. t. a conic, whose focus and directrix are given, passes through a fixed point.

[Hint :—Take the focus as origin and the equation to the directrix as $x = a$; then the equation of the conic is

$$x^2 + y^2 = e^2 (x - a)^2, \text{ where } e \text{ is the eccentricity.}]$$

9. Find the locus of the middle points of the chords of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

which are parallel to the line $lx + my + n = 0$.

The ends of a chord are equidistant from a fixed point (x_0, y_0) . Prove that the locus of the middle point of the chord is the conic

$$(x - x_0)(hx + by + f) - (y - y_0)(ax + hy + g) = 0.$$

(I. C. S. 1941)

10. Find the condition that the lines given by

$$Ax^2 + 2Hxy + By^2 = 0$$

may be parallel to the conjugate diameters of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad (\text{Punjab 1934})$$

11. Show that two concentric conics have in general one and only one pair of common conjugate diameters. (Punjab 1934)

12. Tangents are drawn to the conic $ax^2 + 2hxy + by^2 = 2x$ from two points on the x -axis equidistant from the origin; prove that their four points of intersection lie on the conic

$$by^2 + hxy = x.$$

If the tangents be drawn from two points on the y -axis equidistant from the origin, prove that the points of intersection are on a straight line.

Answers

$$3. \quad \frac{x-x_1}{ax_1+hy_1+g} = \frac{y-y_1}{hx_1+by_1+f}.$$

$$7. \quad \alpha^2 (ax+hy+g)^2 + \beta^2 (hx+by+f)^2 = (gx+fy+c)^2.$$

$$9. \quad m(ax+hy+g) = l(hx+by+f).$$

$$12. \quad hx+by=0.$$

CHAPTER V

PARABOLA AND ELLIPSE (ADVANCED)

5.1. Recapitulation. The following results which have been established in chapters VI and VII, Part I can also be deduced from the general results of Chapter IV.

I. The tangent at (x_1, y_1) on the parabola $y^2 = 4ax$ is

$$yy_1 = 2a(x + x_1). \quad \dots\dots (1)$$

The chord of contact of tangents from an external point (x_1, y_1) and the polar of (x_1, y_1) are also represented by equation (1).

The chord with its middle point as (x_1, y_1) is

$$yy_1 - 2ax = y_1^2 - 2ax_1.$$

The equation of the pair of the tangents from (x_1, y_1) is

$$(y^2 - 4ax)(y_1^2 - 4ax_1) = [yy_1 - 2a(x + x_1)]^2.$$

II. The tangent at (x_1, y_1) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1. \quad \dots\dots (2)$$

The chord of contact of tangents from an external point (x_1, y_1) and the polar of (x_1, y_1) are also represented by equation (2).

The chord with its middle point as (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

The equation of the pair of tangents from (x_1, y_1) is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) \left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - 1 \right) = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - 1 \right)^2.$$

5.2. Co-normal Points on a Parabola. A few more examples on co-normal points are given below.

Ex. 1. Show that the circle through the three points the normals at which to the parabola $y^2 = 4ax$ meet in (h, k) is

$$x^2 + y^2 - (h + 2a)x - \frac{1}{2}ky = 0. \quad (\text{Kashmir '57 ; Agra '60})$$

Let the normals at the points $P(t_1)$, $Q(t_2)$, and $R(t_3)$ of the given parabola meet in (h, k) , then t_1, t_2, t_3 are the roots of

$$at^3 + (2a - h)t - k = 0. \dots\dots (1) \text{ [See § 6.41, Part I.]}$$

Let the equation of the circle through P , Q and R be

$$x^2 + y^2 + 2gx + 2fy + c = 0. \dots\dots (2)$$

The point $(at^2, 2at)$ lies on (2), if

$$a^2t^4 + (4a^2 + 2ga)t^2 + 4fat + c = 0. \dots\dots (3)$$

Since this equation is of the 4th degree in t , it follows that the circle (2) meets the given parabola in four points, *i. e.*, the circle passes through a fourth point, say ' t_4 ', in addition to P , Q and R .

Now, from (1) and (3), we have respectively

$$t_1 + t_2 + t_3 = 0 \quad \text{and} \quad t_1 + t_2 + t_3 + t_4 = 0;$$

$$\therefore t_4 = 0.$$

Hence, the circle PQR passes through the vertex of the parabola and, therefore, $c = 0$. Equation (3) now reduces to

$$at^3 + (4a + 2g)t + 4f = 0, \dots\dots (4)$$

whose roots are t_1, t_2, t_3 . Thus, equations (1) and (4) are identical. Comparing the coefficients in (1) and (4), we have

$$4a + 2g = 2a - h \quad \text{and} \quad 4f = -k,$$

$$\text{i. e.,} \quad 2g = -(h + 2a) \quad \text{and} \quad 2f = -\frac{1}{2}k.$$

Substituting the values of g, f and c in (2), we get

$$x^2 + y^2 - (h + 2a)x - \frac{1}{2}ky = 0, \dots\dots (5)$$

which is the required equation.

Note. The circle (5) is known as *Joachimsthal's circle*.

Ex. 2. Prove that the sum of the angles which the three normals from any point O make with the axis of a parabola exceeds the angle which the focal distance of O makes with the axis by a multiple of π .

Find also the locus of O if two of the normals make complementary angles with the axis. (Vikram 1959)

Let the equation of the parabola be $y^2 = 4ax$, then any normal to it is

$$y = mx - 2am - am^3. \dots\dots (1)$$

This passes through $O(h, k)$, if $k = mh - 2am - am^3$,

$$\text{i. e.,} \quad am^3 + (2a - h)m + k = 0. \dots\dots (2)$$

Since m is the slope of (1), m_1, m_2, m_3 the three roots of (2) are the slopes of the three normals to the parabola from O . Hence, if $m_1 = \tan \theta_1, m_2 = \tan \theta_2, m_3 = \tan \theta_3$, then $\theta_1, \theta_2, \theta_3$ are the angles made by the three normals with the x -axis, the

axis of the parabola. Also, if the focal distance of O makes an angle θ with the x -axis,

$$\text{then } \tan \theta = \frac{k}{h-a}, \quad \dots\dots(3)$$

since the focus of the parabola is $(a, 0)$.

$$\text{From (2), } \Sigma \tan \theta_1 = \Sigma m_1 = 0,$$

$$\Sigma \tan \theta_1 \tan \theta_2 = \Sigma m_1 m_2 = \frac{2a-h}{a}, \quad \dots\dots(4)$$

$$\text{and } \tan \theta_1 \tan \theta_2 \tan \theta_3 = m_1 m_2 m_3 = -\frac{k}{a};$$

$$\begin{aligned} \therefore \tan (\theta_1 + \theta_2 + \theta_3) &= \frac{\Sigma \tan \theta_1 - \tan \theta_1 \tan \theta_2 \tan \theta_3}{1 - \Sigma \tan \theta_1 \tan \theta_2} \\ &= \frac{k/a}{1 - (2a-h)/a} = \frac{k}{h-a} \\ &= \tan \theta, \quad \text{from (3).} \end{aligned}$$

Hence, $\theta_1 + \theta_2 + \theta_3 = n\pi + \theta$, where $n=0, \pm 1, \pm 2, \dots\dots$;
i. e., the sum of the angles made by the three normals from O with the x -axis exceeds the angle made by the focal distance of O with the x -axis by a multiple of π .

If two of the normals make complementary angles with the x -axis, let $\theta_1 + \theta_2 = \pi/2$, say,

$$\text{then } m_1 m_2 = \tan \theta_1 \tan \theta_2 = \tan \theta_1 \tan (\pi/2 - \theta_1) = 1,$$

$$\text{so that } m_3 = -\frac{k}{a}, \text{ from (4).}$$

Since m_3 is a root of (2), we have

$$a\left(-\frac{k}{a}\right)^3 + (2a-h)\left(-\frac{k}{a}\right) + k = 0,$$

$$\text{or } k[k^2 + a(a-h)] = 0.$$

Now $k \neq 0$, for, otherwise $m_3 = 0$ and, therefore, $m_1 + m_2 = 0$ ($\because \Sigma m_1 = 0$) and thus $m_1 m_2$ cannot be equal to 1 if m_1, m_2 are real.

$$\text{Hence, } k^2 + a(a-h) = 0,$$

$$\text{whence the locus of } O \text{ is } y^2 + a(a-x) = 0,$$

$$\text{i. e., } y^2 = a(x-a), \text{ which is a parabola.}$$

* 5.3. Equation of the Parabola $y^2 = 4ax$ referred to Diameter and Tangent at any Point.

Let $P(at^2, 2at)$ be any point on the parabola $y^2 = 4ax$.

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Transferring the origin to P keeping the directions of the axes fixed, the equation of the parabola becomes (§1.8, Part I)

$$(y + 2at)^2 = 4a(x + at^2) \dots (1)$$

If we now transform to another system of axes, keeping the x -axis as before and taking the y -axis as the tangent at P , then the coordinates (x, y) of any point on the parabola (1) referred to rectangular axes through P and the coordinates (X, Y) of the same point referred to the new axes are connected by the relations [§ 3.3]

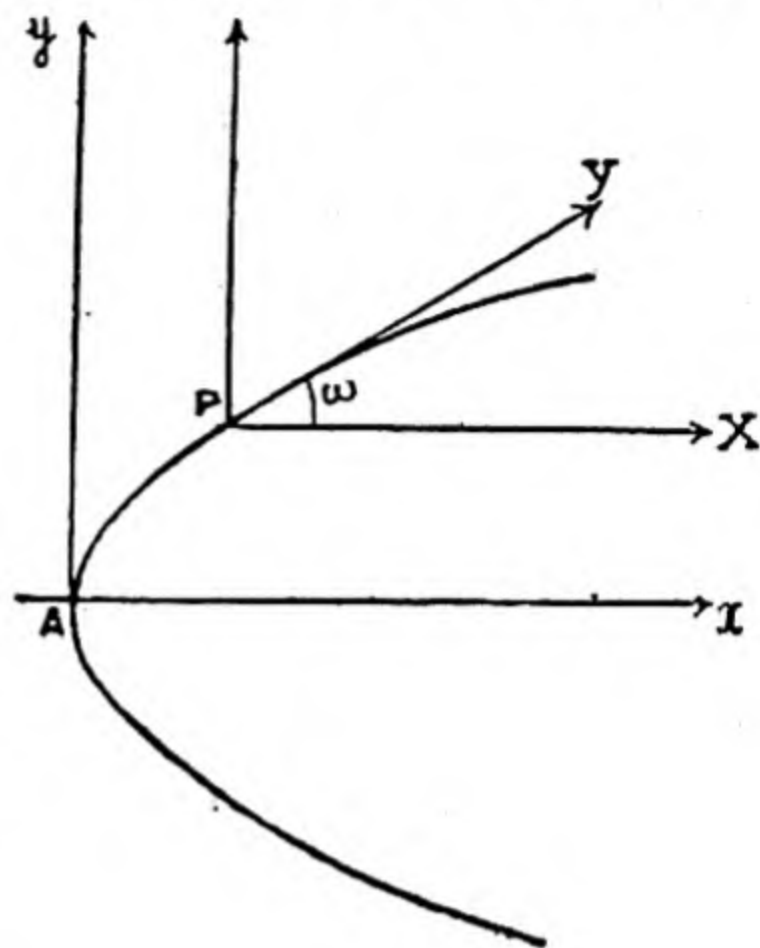


Fig. 9

$$x = X + Y \cos \omega, \quad y = Y \sin \omega, \quad \dots (2)$$

where ω is the angle between the diameter and the tangent at P .

Applying the transformation (2), the equation (1) becomes

$$(Y \sin \omega + 2at)^2 = 4a(X + Y \cos \omega + 2at),$$

$$\text{or} \quad Y^2 \sin^2 \omega + 4aY(t \sin \omega - \cos \omega) = 4aX. \quad \dots (3)$$

Now, $\tan \omega = \text{the slope of the tangent at } P \text{ referred to the original axes}$

$$= \frac{1}{t},$$

$$\text{i.e.,} \quad t = \cot \omega.$$

Substituting this value in (3), we have

$$Y^2 = \frac{4a}{\sin^2 \omega} X, \quad \dots (4)$$

which is of the form $Y^2 = 4bX$,

where $b = a/\sin^2 \omega$.

Hence, the equation of the given parabola when referred to the diameter and tangent at any point is

$$y^2 = 4bx. \quad \dots (5)$$

When $\omega = \pi/2$, the equation (4) reduces to the form $y^2 = 4ax$, which is thus a particular case of the equation (5).

From equation (5), we see that to every value of x there correspond two equal and opposite values of y . It follows,

therefore, that the chords of (5) parallel to the tangent at P (i. e., $x=0$) are bisected by the diameter through P (i. e., $y=0$).

Note. The condition that the line $y=mx+c$ touches the parabola $y^2=4ax$ (see § 6.21, Part I) is obtained by using the fact that the line meets the parabola in two coincident points. This condition does not in any way depend upon the angle between the axes of reference. It follows, therefore, that the $y=mx+c$ touches the parabola (5) above, if $c=\frac{b}{m}$.

Similarly, the equation of the tangent at (x_1, y_1) on the parabola (5) above is $yy_1=2b(x+x_1)$. Further, the equation of the polar of (x_1, y_1) w. r. t. the parabola (5) is $yy_1=2b(x+x_1)$ and the locus of the mid-points of chords parallel to $y=mx$ is $y=\frac{2b}{m}$.

5.4. Co-normal Points on an Ellipse.

The normal at the point ' ϕ ' on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2.$$

If this passes through a given point (h, k) , then

$$ah \sec \phi - bk \operatorname{cosec} \phi = a^2 - b^2.$$

Writing this as

$$\frac{ah \left(1 + \tan^2 \frac{\phi}{2}\right)}{1 - \tan^2 \frac{\phi}{2}} - \frac{bk \left(1 + \tan^2 \frac{\phi}{2}\right)}{2 \tan \frac{\phi}{2}} = a^2 - b^2,$$

and putting $\tan \frac{\phi}{2} = t$, we get on multiplying out

$$bkt^4 + 2(ah + a^2 - b^2)t^3 + 2(ah - a^2 + b^2)t - bk = 0. \quad \dots \dots (1)$$

This equation, being a biquadratic in t , gives four values of t , i. e., of $\tan \frac{\phi}{2}$. If one of the four roots of (1) be t_1 , then

$$\tan \frac{\phi}{2} = t_1 \text{ gives } \phi = 2(n\pi + \tan^{-1} t_1)$$

Since the addition of $2n\pi$ to $2 \tan^{-1} t_1$ gives us again the point ϕ , it follows that to each value of $\tan \frac{\phi}{2}$ there corresponds only one point on the ellipse, real or imaginary.

Hence, four normals (real or imaginary) can be drawn from a given point to an ellipse.

If t_1, t_2, t_3, t_4 be the four roots of (1) corresponding respectively to the eccentric angles $\alpha, \beta, \gamma, \delta$, then

$$\Sigma t_1 \neq \Sigma t_1 t_2 t_3, \quad \Sigma t_1 t_2 = 0, \text{ and } t_1 t_2 t_3 t_4 = -1; \quad \dots\dots (2)$$

$$\therefore \tan \frac{1}{2} (\alpha + \beta + \gamma + \delta) = \frac{\Sigma t_1 - \Sigma t_1 t_2 t_3}{1 - \Sigma t_1 t_2 + t_1 t_2 t_3 t_4} \\ = \infty, \text{ from (2)}$$

or $\frac{1}{2} (\alpha + \beta + \gamma + \delta) = \text{an odd multiple of } \pi/2,$
i. e., $\alpha + \beta + \gamma + \delta = \text{an odd multiple of } \pi.$

Hence, the sum of the eccentric angles of the four co-normal points is an odd multiple of π .

Ex. If α, β, γ be the eccentric angles of three points on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ at which the normals are concurrent, prove that

$$\sin (\beta + \gamma) + \sin (\gamma + \alpha) + \sin (\alpha + \beta) = 0. \quad (\text{Vikram 1960})$$

Let the normals at ' α ', ' β ', ' γ ' meet in (h, k) and let δ be the foot of the fourth normal from (h, k) to the ellipse, then as in § 5.4, we have

$$\Sigma \tan \frac{\alpha}{2} \tan \frac{\beta}{2} = 0, \quad \dots\dots\dots (1)$$

and $\tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \tan \frac{\delta}{2} = -1. \quad \dots\dots\dots (2)$

Eliminating $\tan \frac{\delta}{2}$ from (1) and (2), we get

$$\tan \frac{\alpha}{2} \tan \frac{\beta}{2} + \tan \frac{\alpha}{2} \tan \frac{\gamma}{2} + \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \\ - \cot \frac{\alpha}{2} \cot \frac{\beta}{2} - \cot \frac{\alpha}{2} \cot \frac{\gamma}{2} - \cot \frac{\beta}{2} \cot \frac{\gamma}{2} = 0,$$

or $\Sigma \left(\cot \frac{\alpha}{2} \cot \frac{\beta}{2} - \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \right) = 0, \dots\dots\dots (3)$

where the summation contains the angles α, β and γ only.

Simplifying (3), we have

$$\Sigma \frac{\cos^2 \frac{\alpha}{2} \cos^2 \frac{\beta}{2} - \sin^2 \frac{\alpha}{2} \sin^2 \frac{\beta}{2}}{\sin \frac{\alpha}{2} \cos \frac{\alpha}{2} \sin \frac{\beta}{2} \cos \frac{\beta}{2}} = 0,$$

$$\text{or } \Sigma \frac{\cos \frac{\alpha+\beta}{2} \cos \frac{\alpha-\beta}{2}}{\sin \alpha \sin \beta} = 0,$$

$$\text{or } \Sigma (\cos \alpha + \cos \beta) \sin \gamma = 0,$$

$$\text{or } (\sin \beta \cos \gamma + \sin \gamma \cos \beta) + (\sin \gamma \cos \alpha + \sin \alpha \cos \gamma) + (\sin \alpha \cos \beta + \sin \beta \cos \alpha) = 0,$$

$$\text{or } \Sigma \sin (\beta + \gamma) = 0. \quad \dots\dots (4)$$

Note. Similarly, it can be shown that

$$\sin (\gamma + \delta) + \sin (\delta + \beta) + \sin (\beta + \gamma) = 0, \quad \dots\dots (5)$$

$$\sin (\delta + \alpha) + \sin (\alpha + \gamma) + \sin (\gamma + \delta) = 0, \quad \dots\dots (6)$$

$$\sin (\beta + \delta) + \sin (\delta + \alpha) + \sin (\alpha + \beta) = 0. \quad \dots\dots (7)$$

Adding (4), (5), (6) and (7), we get

$$\Sigma \sin (\alpha + \beta) = 0,$$

where the summation contains all the four angles $\alpha, \beta, \gamma, \delta$.

5.5. Concyclic Points on an Ellipse.

I. *Two straight lines through the points of intersection of an ellipse and any circle make equal angles with the axes.*

The equation of a curve passing through the points of intersection of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\dots\dots (1)$

and the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ $\dots\dots (2)$

is $x^2 + y^2 + 2gx + 2fy + c + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 \right) = 0$. $\dots\dots (3)$

Equation (1) will represent a pair of straight lines, if

$$\begin{vmatrix} 1 + \frac{\lambda}{a^2} & 0 & g \\ 0 & 1 + \frac{\lambda}{b^2} & f \\ g & f & c - \lambda \end{vmatrix} = 0. \quad \dots\dots (4)$$

If this condition is satisfied, then (3) represents a pair of straight lines parallel to the lines

$$x^2 + y^2 + \lambda \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right) = 0,$$

which are of the form $y = \pm mx$. Hence, the two lines make equal angles with the x -axis and also with the y -axis.

Note. Since (4) is a cubic in λ , it follows that in general three pairs of straight lines can be drawn through the points of

intersection of (1) and (2), as can be seen geometrically by drawing a figure. The lines in each pair make equal angles with the axes.

II. To find the coordinates of the centre of the circle through the points on the ellipse (1) whose eccentric angles are α, β , and γ .

Let the eccentric angles of the four points of intersection of the ellipse (1) and the circle (2) be $\alpha, \beta, \gamma, \delta$, then the two

$$\text{lines} \quad \frac{x}{a} \cos \frac{\alpha+\beta}{2} + \frac{y}{b} \sin \frac{\alpha+\beta}{2} = \cos \frac{\alpha-\beta}{2}$$

$$\text{and} \quad \frac{x}{a} \cos \frac{\gamma+\delta}{2} + \frac{y}{b} \sin \frac{\gamma+\delta}{2} = \cos \frac{\gamma-\delta}{2}$$

make equal angles with the x -axis ;

$$\therefore \tan \frac{\alpha+\beta}{2} = - \tan \frac{\gamma+\delta}{2}.$$

$$\text{Hence,} \quad \frac{\alpha+\beta}{2} = n\pi - \frac{\gamma+\delta}{2} \quad (n \text{ being an integer}),$$

$$\text{or} \quad \alpha + \beta + \gamma + \delta = 2n\pi. \quad \dots\dots (5)$$

Since the coordinates of the four points of intersection also satisfy equation (2), the eccentric angles $\alpha, \beta, \gamma, \delta$ satisfy the equation $a^2 \cos^2 \phi + b^2 \sin^2 \phi + 2ga \cos \phi + 2fb \sin \phi + c = 0, \dots (6)$

$$\text{or} \quad [(a^2 - b^2) \cos^2 \phi + 2ga \cos \phi + b^2 + c]^2 = 4f^2 b^2 \sin^2 \phi \\ = 4f^2 b^2 (1 - \cos^2 \phi);$$

$$\therefore \Sigma \cos \alpha = - \frac{4ga}{a^2 - b^2}.$$

Similarly, expressing the equation (6) in terms of $\sin \phi$, we

$$\text{have} \quad \Sigma \sin \alpha = - \frac{4fb}{b^2 - a^2}.$$

As $\delta = 2n\pi - (\alpha + \beta + \gamma)$, it follows that

$$-g = \frac{a^2 - b^2}{4a} [\cos \alpha + \cos \beta + \cos \gamma + \cos (\alpha + \beta + \gamma)]$$

$$\text{and} \quad -f = \frac{b^2 - a^2}{4a} [\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma)].$$

These relations give the coordinates of the circle passing through ' α ', ' β ' and ' γ '.

5.6. Further Propositions on Conjugate Diameters.

I. The area of the parallelogram, formed by the tangents at the extremities of conjugate diameters of an ellipse, is constant.

Let $P'CP$ and $D'CD$ be two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (1), then the coordinates of P and D can be written as $(a \cos \phi, b \sin \phi)$ and $(-a \sin \phi, b \cos \phi)$ respectively (see § 7.93, Part I).

Now, the tangents at P, P' are parallel to $D'CD$ and the tangents at D, D' are parallel to $P'CP$ (see § 7.93, Part I).

Hence, the area of the parallelogram $KLMN$ (fig. 10) formed by the tangents at P, D, P', D' is equal to four times the area of the parallelogram $CPKD$.

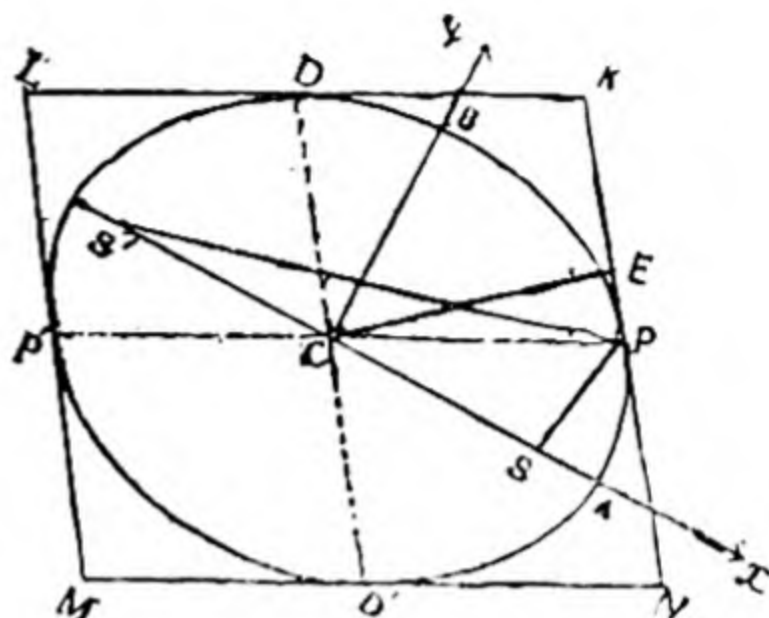


Fig. 10

Draw CE perpendicular upon the tangent at P whose equation is $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1$, then

$$CE = \frac{1}{\sqrt{\frac{\cos^2 \phi}{a^2} + \frac{\sin^2 \phi}{b^2}}} = ab / \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi};$$

\therefore area of the parallelogram $CPKD$

$$= CD \times CE = \sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi} \times \frac{ab}{\sqrt{a^2 \sin^2 \phi + b^2 \cos^2 \phi}} = ab.$$

Thus, the area of the parallelogram $KLMN$ is constant and equal to $4ab$.

Note. The parallelogram $KLMN$ is called the *Conjugate parallelogram*.

II. If CP, CD be two conjugate semi-diameters of an ellipse, then $SP \cdot S'P = CD^2$, where S, S' are the foci of the ellipse.

The coordinates of P and D on the ellipse (1) are $(a \cos \phi, b \sin \phi)$ and $(-a \sin \phi, b \cos \phi)$. Also (see fig. 9),

$$SP = e \left(\frac{a}{e} - a \cos \phi \right) = a - ae \cos \phi,$$

and
$$S'P = e \left(\frac{a}{e} + a \cos \phi \right) = a + ae \cos \phi;$$

$$\begin{aligned}\therefore SP \cdot S'P &= a^2 - a^2 e^2 \cos^2 \phi = a^2 - (a^2 - b^2) \cos^2 \phi \\ &= a^2 \sin^2 \phi + b^2 \cos^2 \phi \\ &= CD^2.\end{aligned}$$

5.61. Equi-Conjugate Diameters.

Definition. Two conjugate diameters of an ellipse, which are equal in length are called *equi-conjugate*.

To find the condition of the equi-conjugate diameters of the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \dots\dots (1)$$

The semi-diameters CP , CD of the ellipse (1) are equi-conjugate, if $CP = CD$,

$$\text{i. e., if} \quad a^2 \cos^2 \phi + b^2 \sin^2 \phi = a^2 \sin^2 \phi + b^2 \cos^2 \phi,$$

$$\text{i. e., if} \quad (a^2 - b^2) \cos 2\phi = 0,$$

$$\text{whence} \quad \cos 2\phi = 0 \quad (\because a \neq b),$$

$$\text{i. e.,} \quad \phi = \frac{\pi}{4} \text{ or } \frac{3\pi}{4}.$$

Taking $\phi = \pi/4$, the equation of CP is

$$y = \frac{b \sin \phi}{a \cos \phi} x = \frac{b}{a} x,$$

and that of CD is

$$y = -\frac{b \cos \phi}{a \sin \phi} x = -\frac{b}{a} x.$$

Hence, the equations of equi-conjugate diameters are

$$y = \pm \frac{b}{a} x.$$

The other value of ϕ , i. e., $3\pi/4$ also gives the same equations.

Note 1. The length of each of the equi-conjugate semi-diameters is $\sqrt{\frac{a^2 + b^2}{2}}$.

Note 2. If tangents are drawn at the extremities of the major and the minor axes, the equations of the diagonals of the rectangle so formed are $y = \pm \frac{b}{a} x$. Hence, the equi-conjugate diameters of an ellipse lie along the diagonals of this rectangle and can be easily constructed.

5.7. Supplemental Chords.

Definition. The chords drawn from any point on an ellipse to the extremities of a diameter are called *supplemental chords*.

To prove that any two supplemental chords of an ellipse are parallel to conjugate diameters.

Let P , any point on an ellipse, be joined to the extremities of a diameter DCD' to form the supplemental chords PD and PD' . Let M and M' be respectively the middle points of PD and PD' (fig. 11), then the lines CM and CM' are respectively parallel to $D'P$ and DP , since C is the middle point of DD' .

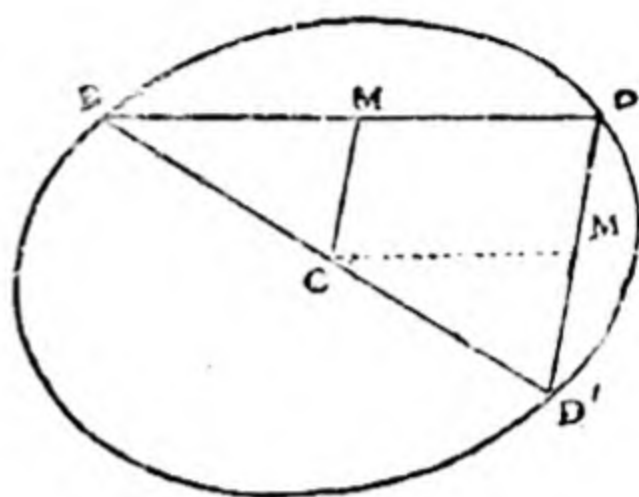


Fig. 11

The diameter along CM , therefore, bisects all chords parallel to DP or CM' , i. e., CM and CM' are portions of conjugate diameters. Hence, the chords PD and PD' are parallel to a pair of conjugate diameters.

*5.8. Equation of an Ellipse referred to Conjugate Diameters.

Let the coordinates of any point on the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad \dots\dots (1)$$

referred to CA, CB as axes be (x, y) and the coordinates of the same point referred to conjugate semi-diameters CP, CD [fig. 10] as axes be (X, Y) , then the two sets of coordinates are connected by the relations $x = lX + mY$, $y = l'X + m'Y$, where l, m, l', m' are constants depending upon the angles ACP and BCD .

The equation (1) when referred to CP, CD as axes, therefore, takes the form $AX^2 + 2HXY + BY^2 = 1$. $\dots\dots (2)$

Since CP , the new X -axis bisects all chords parallel to CD , the new Y -axis, it follows that the points (X_1, Y_1) and $(X_1, -Y_1)$ both lie on (2). Thus,

$AX_1^2 + 2HX_1Y_1 + BY_1^2 = 1$ and $AX_1^2 - 2HX_1Y_1 + BY_1^2 = 1$,
whence $H = 0$.

The equation (2) now reduces to $AX^2 + BY^2 = 1$(3)

If $CP = a'$ and $CD = b'$, then the points $(a', 0)$ and $(0, b')$ both lie on the ellipse (2) ;

$$\therefore Aa'^2 = 1, \text{ i. e., } A = \frac{1}{a'^2},$$

$$\text{and } Bb'^2 = 1, \text{ i. e., } B = \frac{1}{b'^2}.$$

Hence, the equation of an ellipse referred to conjugate diameters of lengths $2a'$, $2b'$ as axes is

$$\frac{X^2}{a'^2} + \frac{Y^2}{b'^2} = 1 \quad \text{.....(4)}$$

Note. If in equation (4), $b' = a'$, we get

$$X^2 + Y^2 = a'^2,$$

which is, therefore, the equation of an ellipse referred to equi-conjugate diameters as axes.

Ex. 1. A circle passes through the end of a diameter of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and also the curve. Prove that the centre of the circle is on the ellipse

$$4a^2x^2 + 4b^2y^2 = (a^2 - b^2)^2.$$

Let $\alpha, \beta, \gamma, \delta$ be the eccentric angles of the four common concyclic points, then $\beta = \alpha + \pi$ and $\gamma = \delta$, say. (1)

Also, $\alpha + \beta + \gamma + \delta = 2n\pi$ (§ 5.5, II) ;

$$\therefore 2\alpha + \pi + 2\gamma = 2n\pi,$$

$$\text{or } \gamma = (2n-1) \frac{\pi}{2} - \alpha. \quad \text{.....(2)}$$

Now, the coordinates of the centre of the circle through the points ' α ', ' β ' and ' γ ' are given by (§ 5.5, II)

$$\begin{aligned} x &= \frac{a^2 - b^2}{4a} [\cos \alpha + \cos \beta + \cos \gamma + \cos (\alpha + \beta + \gamma)] \\ &= (-1)^{n-1} \frac{a^2 - b^2}{2a} \sin \alpha, \end{aligned}$$

$$\begin{aligned} \text{and } y &= \frac{b^2 - a^2}{2a} [\sin \alpha + \sin \beta + \sin \gamma - \sin (\alpha + \beta + \gamma)] \\ &= (-1)^{n-1} \frac{b^2 - a^2}{2a} \cos \alpha, \quad \text{from (1) and (2).} \end{aligned}$$

Eliminating α from these equations, we get

$$4a^2x^2 + 4b^2y^2 = (a^2 - b^2)^2, \quad \text{.....(3)}$$

which is the required locus.

Note. The equation (3) obviously represents an ellipse as can be seen by writing it as

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = \frac{(a^2 - b^2)^2}{4a^2b^2}$$

Ex. 2. Show that the acute angle between two conjugate diameters of an ellipse is least when the conjugate diameters are equal.

Let θ be the acute angle between two conjugate semi-diameters of lengths r_1, r_2 , then (§ 5.6, I)

$$r_1 r_2 \sin \theta = ab ;$$

$\therefore \sin \theta$ is least when $r_1 r_2$ is greatest.

Now, $r_1^2 + r_2^2 = a^2 + b^2 = \text{constant}$ [see §7.93, Part I]

and $2r_1 r_2 = (r_1^2 + r_2^2) - (r_1 - r_2)^2 ;$

$\therefore r_1 r_2$ is greatest when $r_1 = r_2$.

Hence, θ is least when $r_1 = r_2$.

Ex. 3. Prove that the locus of the intersection of normals at the extremities of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

is the curve $2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2.$

(Allahabad '58 ; Agra '59)

Let ϕ and $\phi + \pi/2$ be the eccentric angles of the two extremities of a pair of conjugate diameters, then the normals at these points are

$$-\frac{ax}{\cos \phi} - \frac{by}{\sin \phi} = a^2 - b^2, \quad \dots \dots (1)$$

$$\text{and} \quad -\frac{ax}{\sin \phi} - \frac{by}{\cos \phi} = a^2 - b^2. \quad \dots \dots (2)$$

Eliminating ϕ from (1) and (2), we get the required locus. Solving (1) and (2) for $\frac{1}{\cos \phi}$ and $\frac{1}{\sin \phi}$, we have

$$\frac{a^2x^2 + b^2y^2}{\cos \phi} = (a^2 - b^2)(ax - by),$$

$$\text{and} \quad \frac{a^2x^2 + b^2y^2}{\sin \phi} = -(a^2 - b^2)(ax + by).$$

Hence, the required equation is

$$\left(\frac{a^2x^2 + b^2y^2}{a^2 - b^2} \right)^2 \left[\left(\frac{1}{ax - by} \right)^2 + \frac{1}{(ax + by)^2} \right] = 1,$$

$$\text{or} \quad 2(a^2x^2 + b^2y^2)^3 = (a^2 - b^2)^2(a^2x^2 - b^2y^2)^2.$$

Exercise 5

1. Find the locus of a point when the three normals drawn from it to the parabola $y^2=4ax$ are such that

- (i) one bisects the angle between the other two ;
- (ii) the area of the triangle formed by their feet is constant.

2. If the normals at three points P, Q, R , on the parabola $y^2=4ax$ meet at a point O , and S be the focus, prove that

$$SP.SQ.SR=a.OS^2. \quad (\text{Kashmir 1951})$$

3. Prove that the normals at the points, where the line $lx+my=1$ meets the parabola $y^2=4ax$, meet on the normal at the point $\left(\frac{4am^2}{l^2}, \frac{4am}{l}\right)$ of the parabola. (Aligarh 1951)

4. If the normals at P, Q, R on the parabola $y^2-4ax=0$ meet in the point (h, k) , then the orthocentre of the triangle PQR will be $\left(h-6a, -\frac{k}{2}\right)$.

Prove also that the centroid of PQR is $\left[\frac{2}{3}(h-2a), 0\right]$.

5. From any point (h, k) three normals are drawn to the parabola $y^2-4ax=0$, and the tangents at their feet are drawn. Prove that the coordinates of the vertices of the triangle formed by these tangents are given by

$$x^3+(h-2a)x^2-ak^2=0,$$

and

$$y^3-a(h-2a)y+a^2k=0.$$

6. The normals at P, Q, R on the parabola $y^2=4ax$ meet at a point on the line $x=a$. Prove that the sides of the triangle PQR touch the parabola

$$y^2=16a(x+2a-a). \quad (\text{Utkal 1947})$$

[**Hint** :—If the points P, Q, R are $(am_r^2, -2am_r)$,

$$r=1, 2, 3; \text{ then } m_3^2-m_1m_2=\frac{a}{a}-2, \quad \dots\dots (1)$$

after simplification. Also, the equation of PQ is

$$2x+(m_1+m_2)y+2am_1m_2=0,$$

$$\text{or } 2x-m_3y+2a\left(m_3^2-\frac{a}{a}+2\right) \quad (\because \text{ of (1) and } \Sigma m_1=0)$$

$$\text{or } y=\lambda(x+2a-a)-\frac{4a}{\lambda}, \quad \dots\dots (2)$$

where $\lambda = \frac{2}{m_3}$.

Now, $y = \lambda (x - h) \pm \frac{a}{\lambda}$ always touches the parabola $y^2 = 4a (x - h)$;

\therefore (2) always touches the parabola $y^2 = 16a (x + 2a - a)$.]

7. The normals at the points whose eccentric angles are $\alpha, \beta, \gamma, \delta$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet at a point.

Prove that $\Sigma \cos (\alpha + \beta) = 0$. (Lucknow 1951)

8. If the normals at the four points $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ and (x_4, y_4) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ are concurrent, prove that :

$$(i) \quad \Sigma x_1 \cdot \Sigma \frac{1}{x_1} = 4. \quad (\text{Delhi 1947})$$

$$(ii) \quad \Sigma y_1 \cdot \Sigma \frac{1}{y_1} = 4.$$

[Note. This problem can also be stated as follows :

If the normals at the points on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ whose eccentric angles are $\alpha, \beta, \gamma, \delta$ be concurrent, then

$$(i) \quad \Sigma \cos \alpha \cdot \Sigma \sec \alpha = 4.$$

$$(ii) \quad \Sigma \sin \alpha \cdot \Sigma \operatorname{cosec} \alpha = 4.]$$

9. The normals at four points on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet in the point (α, β) . Prove that the mean position of the four points is

$$\left\{ \frac{a^2 \alpha}{2(a^2 - b^2)}, \frac{b^2 \beta}{2(b^2 - a^2)} \right\}. \quad (\text{Lucknow 1944})$$

10. Show that the condition that the normals at $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ may be concurrent is

$$\begin{vmatrix} x_1 & y_1 & x_1 y_1 \\ x_2 & y_2 & x_2 y_2 \\ x_3 & y_3 & x_3 y_3 \end{vmatrix} = 0.$$

11. If three points are taken on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ such that their centroid has a fixed position (h, k) , the centre of the circle passing through them lies on the curve

$$\left(4ax - \frac{3hc^2}{a}\right)^2 + \left(4by + \frac{3kc^2}{b}\right)^2 = c^4,$$

where $c^2 = a^2 - b^2$.

12. Prove that the locus of the centre of a circle which cuts the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the fixed point (α, β) and in two other points at the extremities of a diameter is the ellipse

$$2a^2x^2 + 2b^2y^2 = (a^2 - b^2)(\alpha x - \beta y). \quad (\text{Utkal 1947})$$

13. Show that the locus of the centroid of an equilateral triangle inscribed in $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{x^2}{a^2}(a^2 + 3b^2)^2 + \frac{y^2}{b^2}(b^2 + 3a^2)^2 = (a^2 - b^2)^2.$$

14. In an ellipse a pair of conjugate diameters is produced to meet the directrix. Show that the orthocentre of the triangle so formed is the focus. (Nagpur 1947)

15. Prove that if CP and CQ be two conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, the locus of the middle point of

$$PD \text{ is } \frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{2}. \quad (\text{Delhi 1958})$$

16. CP, CQ are conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and the circles with CP and CQ as diameters intersect at R ; show that R lies on the curve

$$2(x^2 + y^2)^2 = a^2x^2 + b^2y^2. \quad (\text{Punjab 1956})$$

Note. Since $\angle CRP$ and $\angle CRQ$ are both right angles being angles in a semi-circle, it follows that R is the foot of the perpendicular from C upon PQ . Hence, the problem can also be stated as :

If CP and CQ are conjugate semi-diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, show that the locus of the foot of the perpendicular from C upon PQ is

$$2(x^2 + y^2)^2 = a^2x^2 + b^2y^2. \quad (\text{Aligarh 1951})$$

17. If λ, λ' be the angles which any two conjugate diameters subtend at any fixed point on an ellipse, prove that $\cot^2 \lambda + \cot^2 \lambda'$ is constant. (*Allahabad 1952*)

18. If two conjugate semi-diameters CP and CQ of an ellipse cut the director circle in A and B , prove that the straight line AB touches the ellipse. (*Patna 1958*)

19. Show that the lines $Ax^2 + 2Hxy + By^2 = 0$ are conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, if $Aa^2 + Bb^2 = 0$.

20. Show that two parallel tangents to an ellipse are cut by any other tangent in points which lie on conjugate diameters (*Travancore 1957*)

21. If p be the length of the perpendicular from the centre of an ellipse upon the tangent at any point P on it and r the distance of P from the centre, show that

$$\frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} - \frac{r^2}{a^2 b^2}.$$

22. Two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet a fixed straight line $lx + my = 1$ in P and Q and the straight lines through P, Q perpendicular to these diameters intersect in R ; prove that the locus of R is the straight line

$$a^2 lx + b^2 my = a^2 + b^2.$$

23. If the points of intersection of the ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ and } \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} = 1$$

be the end-points of the conjugate diameters of the former, prove that $\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} = 2$. (*Agra 1956*)

24. If the semi-diameter CQ of an ellipse be conjugate to the normal at P , prove that CP will be conjugate to the normal at Q .

25. If P, D be the extremities of conjugate diameters of an ellipse, and PP', DD' be chords parallel to an axis of the ellipse; show that PD' and $P'D$ are parallel to the equi-conjugates,

26. If P, D are the extremities of conjugate diameters of an ellipse, and the tangents at P, D cut respectively the major and minor axes in T, T' ; show that TT' will be parallel to one of the equi-conjugates.

27. If any pair of conjugate diameters of an ellipse cut the tangent at a point P in T and T' , show that

$$TP \cdot PT' = CD^2,$$

where CD is the diameter conjugate to CP . (Punjab 1940)

[**Hint** :—Take CP and CD as coordinate axes.]

28. The tangent at any point P of an ellipse cuts the equi-conjugate diameters in T, T' ; show that the triangles $TCP, T'CP$ are in the ratio of $CT^2 : CT'^2$, where C is the centre of the ellipse.

[**Hint** :—Take the equi-conjugate diameters as axes.]

Answers

1. (i) $27ay^2 = (5a - x)^2 (2x - a)$; (ii) $4(x - 2a)^3 - 27ay^2 =$
constant.

CHAPTER VI

THE HYPERBOLA

6.1. Definition. A *hyperbola* is a conic whose eccentricity e is >1 (§ 6.1, Part I).

Thus, a hyperbola may be defined as the locus of a point which moves so that the ratio of its distance from a fixed point (called the focus) and from a fixed line (called the directrix) is a constant greater than unity.

6.11. Equation of a Hyperbola.

I. General Equation.

From the definitions of an ellipse and a hyperbola, it is clear that the only difference between the two pertains to the value of eccentricity. Hence, the general equation of the hyperbola is the same as that of the ellipse (see § 7.11, Part I). However, for the hyperbola $e > 1$, and $B^2 - AC > 0$.

Note. A necessary condition that the general equation of the second degree $Ax^2 + 2Bxy + Cy^2 + Dx + Ey + F = 0$ represents a hyperbola is that $B^2 - AC > 0$. For a detailed discussion of this condition, see chapter VII.

II. Standard Equation.

Let S be the focus of a hyperbola (eccentricity e), SZ the perpendicular from S on its directrix DZ and A the point on SZ such that $AS = e \cdot AZ$, then, by definition, A lies on the hyperbola. Since $e > 1$, there will be another point A' on SZ produced such that $A'S = e \cdot A'Z$, i. e., the points A and A' divide SZ internally and externally in the ratio

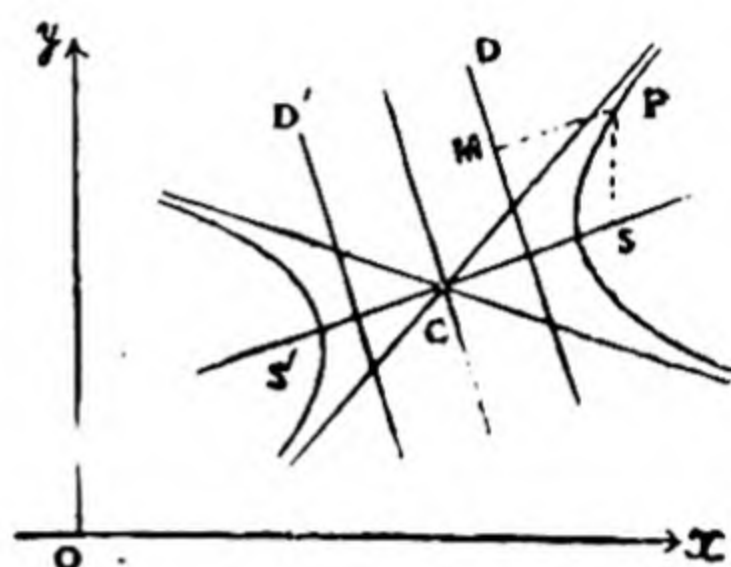


Fig. 12

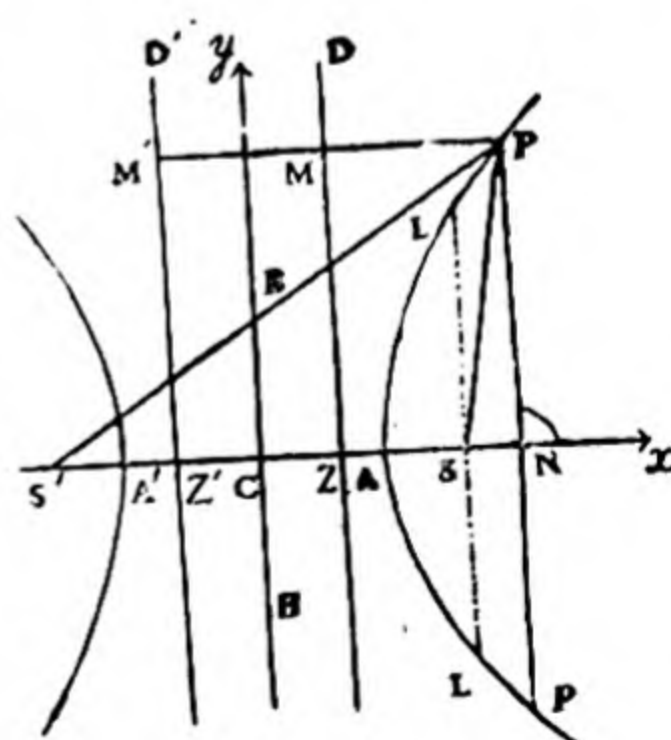


Fig. 13

$e > 1$. The point A' also lies on the ellipse. Let $AA' = 2a$ and let C be the middle point of AA' , so that $A'C = CA = a$ and

$$\begin{aligned} 2a &= A'A = A'S - AS \\ &= e (A'Z - ZA) \\ &= 2e \cdot CZ; \end{aligned}$$

$$\therefore CZ = \frac{a}{e}. \quad \dots\dots (1)$$

$$\begin{aligned} \text{Also, } A'S + AS &= e (A'Z + ZA) = e \cdot AA', \\ \text{or } (a + CS) + (CS - a) &= e \cdot 2a; \\ \therefore CS &= ae. \quad \dots\dots (2) \end{aligned}$$

Now, let C be the origin, CA' the x -axis and a line perpendicular to it as the y -axis, then S is the point $(ae, 0)$ and the directrix is the straight line $x = a/e$.

Let $P(x, y)$ be any point on the hyperbola, then if PN be the ordinate of P and PM perpendicular to the directrix,

$$MP = ZN = CN - CZ = x - \frac{a}{e}.$$

Since $SP = e \cdot PM$, i. e., $SP^2 = e^2 \cdot PM^2$, by definition ;

$$\begin{aligned} \therefore (x - ae)^2 + y^2 &= e^2 \left(x - \frac{a}{e} \right)^2 \\ \text{or } (e^2 - 1) x^2 - y^2 &= a^2 (e^2 - 1), \\ \text{or } \frac{x^2}{a^2} - \frac{y^2}{a^2 (e^2 - 1)} &= 1, \\ \text{or } \frac{x^2}{a^2} - \frac{y^2}{b^2} &= 1, \quad \dots\dots (3) \end{aligned}$$

where $b^2 = a^2 (e^2 - 1) > 0$, ($\because e > 1$). Thus, b is real.

6.12. Some Definitions and Properties of the Hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \quad \dots\dots (1)$$

From equation (1), we observe the following particulars about the hyperbola (see § 7.12, Part I) :

(i) The hyperbola, like the ellipse, is symmetrical about both the axes and is a central curve with centre at $C(0, 0)$.

(ii) The x -axis meets the hyperbola in the points $A'(-a, 0)$ and $A(a, 0)$, which are called the *vertices* of the hyperbola. The straight line AA' is called the *transverse axis*.

(iii) If $x=0$, then $\frac{y^2}{b^2} = -1$, i. e., the y -axis meets the hyperbola in imaginary points.

However, the line joining $B(0, b)$ and $B'(0, -b)$ on the y -axis is called the *conjugate axis*. Obviously B, B' do not lie on the hyperbola.

(iv) If $-a < x < a$, then $y^2 < 0$, i. e., y is imaginary. No part of the curve, therefore, lies between the lines $x=-a$ and $x=a$. For large value of $|x| > a$, $|y|$ also has large values and as $|x| \rightarrow \infty$, $|y|$ also $\rightarrow \infty$.

The curve, therefore, consists of two branches each extending to infinity on either side and is as shown in fig. 13.

(v) As in the case of the ellipse, there is a second focus $S'(-ae, 0)$ and a corresponding second directrix $D'Z'$ whose equation is $x = -a/e$.

(vi) The double ordinate PNP' of a point P on the hyperbola is the chord through P perpendicular to the transverse axis and the *latus-rectum* (i. e., a double ordinate through either focus) is of length $\frac{2b^2}{a}$ as in the case of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(vii) The circle on AA' as diameter is called the *auxiliary circle* of the hyperbola.

Note. The transverse axis and the conjugate axis are jointly called the *principal axes*.

6.13. A Geometrical Property of a Hyperbola.

In fig. 13, the coordinates of a point P on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

are (CN, PN) ; $\therefore \frac{CN^2}{a^2} - \frac{PN^2}{b^2} = 1,$

or $\frac{PN^2}{CB^2} = \frac{CN^2}{CA^2} - 1 = \frac{CN^2 - CA^2}{CA^2} = \frac{AN \cdot A'N}{CA^2},$

or $\frac{PN^2}{AN \cdot A'N} = \frac{CB^2}{CA^2} \dots \dots (1)$

6.14. The Focal Distances of a Point.

In fig. 13, if $P(x, y)$ is any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

then $SP = e$. $MP = e (CN - CZ) = ex - a$,
 and $S'P = e$. $M'P = e (Z'C + CN) = a + ex$;
 $\therefore S'P - SP = 2a = AA'$ (1)

i. e., the difference of the focal distances of a point on the hyperbola is equal to the transverse axis.

6.15. The point $P (x_1, y_1)$ lies outside or inside the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, according as the expression $\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1$ is positive or negative.

The proof is similar to that of § 6.15, Part I and is left as an exercise to the reader.

6.16. Limiting Case of the Equation $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (1)

Writing (1) as $x^2 - \frac{y^2}{e^2 - 1} = a^2$, [$\because b^2 = a^2(e^2 - 1)$],

we see that if $a \rightarrow 0$ while e remains constant, then the limiting form of the equation is

$$x^2 - \frac{y^2}{1 - e^2} = 0,$$

which represents a pair of straight lines through the origin and equally inclined to the axes.

Thus, a pair of straight lines is the limiting case of a hyperbola whose axes tend to zero while their ratio remains finite.

6.17. Parametric Representation.

The coordinates of any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots\dots(1)$$

can be represented by $x = a \sec \theta$, $y = b \tan \theta$,(2)

where θ is a parameter, as can be seen by actual substitution. The point $(a \sec \theta, b \tan \theta)$ is often referred to as the point ' θ '.

This parameter θ is not so important as the eccentric angle of a point on an ellipse and there is no special name for it. Geometrically, it can be defined as follows.

Let $P(x, y)$ be a point on the hyperbola (1). Now, draw the ordinate PN and also the tangent NE to touch the auxiliary circle at E (fig. 14); then in the right-angled triangle CEN ,

$$CN = CE \sec \widehat{NCE},$$

i. e., $x = a \sec \widehat{NCE} \dots \dots (3)$

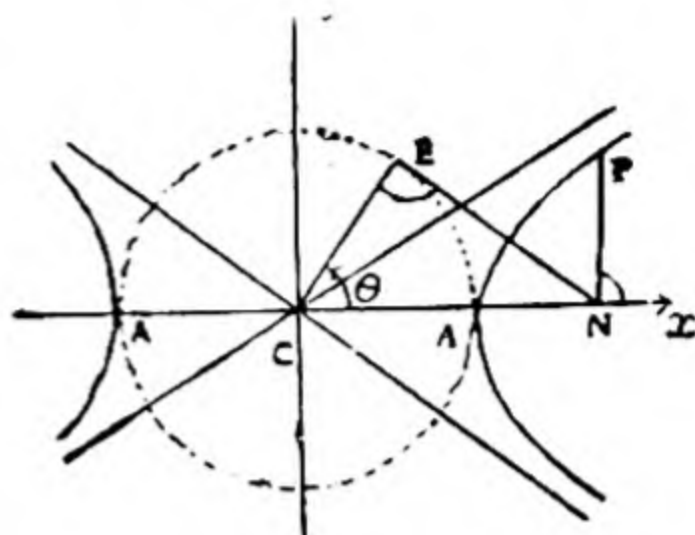


Fig. 14

$$\text{Also, } y = b \sqrt{\frac{x^2}{a^2} - 1}$$

$$= b \tan \widehat{NCE}, \text{ from (3).}$$

Thus, \widehat{NCE} is the parameter θ of the point P .

Note. Another parametric representation which is not commonly used is $x = a \cosh \theta, y = b \sinh \theta \dots \dots (4)$

The equations (2) and (4) are called the *freedom equations* of the hyperbola (1).

6.2. Propositions about the Hyperbola.

$$\text{Since the equation } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \dots \dots (1)$$

$$\text{differs from the equation } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \dots \dots (2)$$

only in the sign of b^2 , many of the propositions for the ellipse (2) hold good for the hyperbola (1) when b^2 is changed to $-b^2$.

Some of the results for the hyperbola (1) are enumerated below. These results can be proved independently as for the general conic in Chapter IV or derived from similar results for the ellipse.

(i) The straight line $y = mx + c$ is a tangent to the hyperbola if $c^2 = a^2 m^2 - b^2$, i. e., the lines $y = mx \pm \sqrt{a^2 m^2 - b^2}$ always touch the hyperbola for all values of m .

(ii) The tangent at a point (x_1, y_1) on the hyperbola is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1.$$

This equation also represents the chord of contact of tangents drawn from (x_1, y_1) and the polar of (x_1, y_1) .

(iii) The normal at a point (x_1, y_1) on the hyperbola is

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{-y_1/b^2}.$$

(iv) The equation of the pair of tangents from (x_1, y_1) is

$$SS_1 = T^2,$$

where $S \equiv \frac{x^2}{a^2} - \frac{y^2}{b^2} - 1, \quad S_1 \equiv \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1,$

and $T \equiv \frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1.$

(v) The equation of the director circle is

$$x^2 + y^2 = a^2 - b^2,$$

which is real or imaginary according as $b^2 < a^2$ or $b^2 > a^2$. If $b^2 = a^2$, the director circle reduces to a point circle.

(vi) The equation of the chord with its middle point as (x_1, y_1) is

$$T = S_1,$$

where T and S_1 have the same meaning as in (iv), above.

(vii) If the polar of a point P passes through Q , then the polar of Q passes through P . The points P and Q are called *conjugate points* w. r. t. the hyperbola.

(viii) If the pole of a straight line AB lies on CD , then the pole of CD lies on AB . The straight lines AB and CD are called *conjugate lines* w. r. t. the hyperbola.

(ix) The locus of the mid-points of a system of parallel chords of a hyperbola is a straight line through the centre of the hyperbola which is called a *diameter* of the hyperbola.

The chords parallel to a diameter $y = mx$ are bisected by the diameter $y = m'x$, where $mm' = \frac{b^2}{a^2}$. The two diameters are called *conjugate diameters*.

Note. The conjugate diameters are conjugate lines w. r. t. the hyperbola.

6.3. Tangent and Normal at the Point ' θ '.

Putting $x_1 = a \sec \theta$ and $y_1 = b \tan \theta$ in the equation

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1,$$

we see that the tangent at the point $(a \sec \theta, b \tan \theta)$ on the

hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1, \quad \dots\dots (1)$$

or
$$\frac{x}{a} - \frac{y}{b} \sin \theta = \cos \theta. \quad \dots\dots (2)$$

The normal at the point ' θ ' is

$$y - b \tan \theta = - \frac{a \tan \theta}{b \sec \theta} (x - a \sec \theta),$$

or
$$ax \cos \theta + by \cot \theta = a^2 + b^2. \quad \dots\dots (3)$$

Note. The equation (1) can be independently obtained as follows : The equation of the chord joining the points ' θ ' and ' ϕ ' on the hyperbola is

$$\begin{aligned} \frac{y - b \tan \theta}{x - a \sec \theta} &= \frac{b (\tan \theta - \tan \phi)}{a (\sec \theta - \sec \phi)} = \frac{b \sin (\theta - \phi)}{a (\cos \phi - \cos \theta)} \\ &= \frac{b}{a} \frac{\cos \frac{\theta - \phi}{2}}{\sin \frac{\theta + \phi}{2}}. \end{aligned}$$

when $\phi \rightarrow \theta$, the equation of the tangent at the point ' θ ' is

$$\frac{y - b \tan \theta}{x - a \sec \theta} = \frac{b}{a \sin \theta},$$

i. e.,
$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1.$$

6.31. Co-normal Points. In § 6.3 (3), writing $\tan \frac{\theta}{2} = t$, we get a biquadratic in t which shows that four normals can be drawn to a hyperbola from a given point.

Ex. 1. A series of chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are tangents to the circle described on the straight line joining the foci of the hyperbola as diameter ; show that the locus of their poles w. r. t. the hyperbola is

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2} \quad \begin{array}{l} \text{(Rajputana 1957)} \\ \text{(Allahabad 1958)} \end{array}$$

The equation of the circle on the line joining $(ae, 0)$ and $(-ae, 0)$ as diameter is

$$x^2 + y^2 = a^2 e^2 = a^2 + b^2. \quad \dots\dots(1)$$

Now, the polar of (x', y') w. r. t. the hyperbola is

$$\frac{xx'}{a^2} - \frac{yy'}{b^2} = 1. \quad \dots\dots(2)$$

If this is a tangent to the circle (1), then

$$\frac{x'^2}{a^4} + \frac{y'^2}{b^4} = \frac{1}{a^2 + b^2}.$$

Hence, the locus of the pole of (2) w. r. t. the hyperbola

is
$$\frac{x^2}{a^4} + \frac{y^2}{b^4} = \frac{1}{a^2 + b^2}.$$

Ex. 2. Prove that the locus of the middle points of chords of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passing through the fixed point (h, k) is a

hyperbola whose centre is the point $\left(\frac{1}{2}h, \frac{1}{2}k\right)$. (Vikram 1960)

The equation of a chord of the hyperbola whose middle point is (x_1, y_1) is

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}. \quad \dots\dots(1)$$

If it passes through the fixed point (h, k) , then

$$\frac{hx_1}{a^2} - \frac{ky_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}.$$

Hence, the locus of (x_1, y_1) is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{hx}{a^2} - \frac{ky}{b^2},$$

or
$$\frac{(x - \frac{1}{2}h)^2}{a^2} - \frac{(y - \frac{1}{2}k)^2}{b^2} = \frac{h^2}{4a^2} - \frac{k^2}{4b^2},$$

which is a hyperbola with its centre at $(\frac{1}{2}h, \frac{1}{2}k)$.

Ex. 3. If any tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets its director circle in P and Q , show that CP, CQ , are along a pair of conjugate diameters of the hyperbola, where C is its centre.

The tangent at any point $(a \sec \theta, b \tan \theta)$ on the hyperbola

is
$$\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1.$$

If it meets the director circle $x^2 + y^2 = a^2 - b^2$ in the points P, Q , then the joint equation of the straight lines CP, CQ is

$$(x^2 + y^2) = (a^2 - b^2) \left(\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta \right)^2,$$

$$\text{or } \left(\frac{a^2 - b^2}{a^2} \sec^2 \theta - 1 \right) x^2 - \frac{2}{ab} (a^2 - b^2) \sec \theta \tan \theta \cdot xy \\ + \left(\frac{a^2 - b^2}{b^2} \tan^2 \theta - 1 \right) y^2 = 0,$$

$$\text{or } b^2 (a^2 \tan^2 \theta - b^2 \sec^2 \theta) x^2 - 2ab (a^2 - b^2) \sec \theta \tan \theta \cdot xy \\ + a^2 (a^2 \tan^2 \theta - b^2 \sec^2 \theta) y^2 = 0; \dots\dots(1)$$

\therefore if the slopes of the lines CP , CQ be m , m' then

$$m m' = \frac{\text{coeff. of } x^2 \text{ in (1)}}{\text{coeff. of } y^2 \text{ in (1)}} = \frac{b^2}{a^2}.$$

Hence, CP and CQ are along conjugate diameters of the hyperbola.

6.4. Asymptotes. A straight line which meets a conic in two points at infinity, but itself lies at a finite distance from the origin, is called an *asymptote* of the conic.

To find the equation of the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \dots\dots(1)$$

$$\text{Let the straight line } y = mx + c \dots\dots(2)$$

be an asymptote of the hyperbola (1).

The x -coordinates of the points of intersection of (1) and (2) are given by

$$\frac{x^2}{a^2} - \frac{(mx + c)^2}{b^2} = 1, \\ \text{or } \left(\frac{1}{a^2} - \frac{m^2}{b^2} \right) x^2 - \frac{2mc}{b} x - \left(\frac{c^2}{b^2} + 1 \right) = 0. \dots\dots(3)$$

Since (2) is an asymptote, both the roots of (3) are infinite ;

$$\therefore \frac{1}{a^2} - \frac{m^2}{b^2} = 0, \text{ and } \frac{2mc}{b} = 0,$$

$$\text{whence } m = \pm \frac{b}{a} \text{ and } c = 0.$$

Hence, the two asymptotes of the hyperbola (1) are given

$$\text{by } y = \frac{b}{a}x \quad \text{and} \quad y = -\frac{b}{a}x,$$

$$\text{i. e., } \frac{x}{a} \pm \frac{y}{b} = 0, \dots\dots(4)$$

$$\text{or } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0, \dots\dots(5)$$

on combining the two equations.

Note 1. The equation of the pair of tangents from the centre to the hyperbola (1) is obtained by putting $x_1=0, y_1=0$ in

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 1\right) \left(\frac{x_1^2}{a^2} - \frac{y_1^2}{b^2} - 1\right) = \left(\frac{xx_1}{a^2} - \frac{yy_1}{b^2} - 1\right)^2$$

and, therefore, is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

Thus, we see that *the asymptotes of a hyperbola are the pair of tangents drawn from its centre.*

Note 2. An asymptote of a curve is also *defined* as a straight line (in the finite plane) which is the limiting position of a tangent to the curve when the point of contact tends to infinity.

The tangent at $(a \sec \theta, b \tan \theta)$ on (1) is [§ 6.3 (2)]

$$\frac{x}{a} - \frac{y}{b} \sin \theta = \cos \theta. \quad \dots\dots(6)$$

As $\theta \rightarrow \pm \frac{\pi}{2}$, the point ' θ ' tends to infinity and the limiting form of the equation (6) is $\frac{x}{a} - \frac{y}{b} = 0$ or $\frac{x}{a} + \frac{y}{b} = 0$, which represent the two asymptotes.

Thus, *the hyperbola approaches indefinitely near the asymptotes in the neighbourhood of infinity.* This fact is of great help in plotting the curve.

Note 3. In the case of an ellipse and a circle which are closed curves it is evident that the point of contact of a tangent can never tend to infinity and thus there cannot be asymptotes. Analytically, it can be shown that the asymptotes of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and the circle $x^2 + y^2 = a^2$ are respectively given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$ and $x^2 + y^2 = 0$, each of which represents a pair of imaginary straight lines, through the origin.

In the case of the parabola $y^2 = 4ax$, the line $y = mx + \frac{a}{m}$ always touches the parabola at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$. As $m \rightarrow 0$, the point of contact tends to infinity and the limiting position of $y = mx + \frac{a}{m}$ lies wholly at infinity. Hence, there is no

asymptote of a parabola in the finite plane. In fact, a parabola has a pair of coincident asymptotes lying wholly at infinity.

6.41. Geometrical Construction of Asymptotes.

If the rectangle KL
 MN (fig. 15) is formed by drawing straight lines through A, A', B, B' , parallel to the principal axes of the hyperbola

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, then the equations of the diagonals

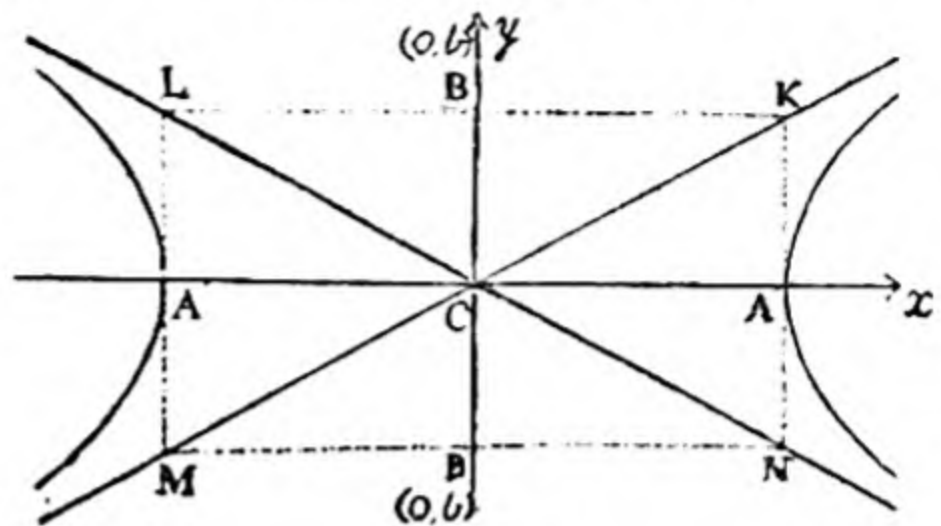


Fig. 15

KM, LN are $y = \frac{b}{a}x$ and $y = -\frac{b}{a}x$ respectively.

Hence, the asymptotes of the hyperbola lie along the diagonals of the rectangle $KL MN$.

6.5. Conjugate Hyperbola. If two concentric hyperbolas are such that the transverse axis of each is the conjugate axis of the other, then each is called the *conjugate hyperbola* of the other and the two hyperbolas are called conjugate hyperbolas.

Thus, the conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (1)

is $\frac{y^2}{b^2} - \frac{x^2}{a^2} = 1$, i. e., $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$, (2)

since an interchange of axes means an interchange of coordinates.

Note 1. It can be easily shown that the asymptotes of (2)

are $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$ (3)

Thus, a hyperbola and its conjugate have the same asymptotes.

Note 2. Again, in equations (1), (2) and (3), we observe that the left hand members are the same, whereas the right hand members of (1) and (3) differ by a constant and the right hand members of (3) and (2) differ by the same constant. This relation among the three equations remains undisturbed when the axes are changed in any manner. Hence, when the general equation of the second degree represents a hyperbola, the equation of the asymptotes differs from that of the hyperbola by a

constant and the equation of the conjugate hyperbola differs from that of the asymptotes by the same constant.

Also see Ch. VII

6.6. Propositions on Conjugate Diameters and Conjugate Hyperbolas.

I. *If a pair of diameters be conjugate w. r. t. a hyperbola, they will be conjugate w. r. t. its conjugate hyperbola.*

The diameters $y=mx$ and $y=m'x$ are conjugate w. r. t. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, (1)

if $mm' = \frac{b^2}{a^2}$ (2)

Now, the equation of the conjugate hyperbola is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1, \text{ i. e., } \frac{x^2}{-a^2} - \frac{y^2}{-b^2} = 1, \text{ (3)}$$

and the above diameters are conjugate w. r. t. it, if

$$mm' = \frac{-b^2}{-a^2} = \frac{b^2}{a^2},$$

which condition is the same as (2). Hence, the result.

II. *If any diameter of a hyperbola meets it in real points, then it will meet the conjugate hyperbola in imaginary points and the conjugate diameter will meet the conjugate hyperbola in real points.*

The equation of a diameter of the hyperbola (1) through any point $P(a \sec \theta, b \tan \theta)$ is

$$y = \frac{b \tan \theta}{a \sec \theta} x, \text{ i. e., } \frac{y}{b} = \frac{x}{a} \sin \theta. \text{ (4)}$$

The x -coordinates of the points where (4) meets (3) are given by $\frac{x^2}{a^2} (1 - \sin^2 \theta) = -1$, i. e., $x^2 = -a^2 \sec^2 \theta$, (5)

whence the two values of x are imaginary. Hence, (4) meets the conjugate hyperbola in imaginary points.

Again, the equation of the diameter conjugate to (4) w. r. t.

$$(1) \text{ is } y = \frac{b}{a \sin \theta} x, \text{ i. e., } \frac{y}{b} = \frac{x}{a} \operatorname{cosec} \theta, \text{ (6)}$$

since the product of the slopes of conjugate diameters is $\frac{b^2}{a^2}$.

The x -coordinates of the points of intersection of (6) and

(3) are given by $\frac{x^2}{a^2}(1 - \operatorname{cosec}^2 \theta) = -1$, i. e., $x^2 = a^2 \tan^2 \theta$,

whence $x = a \tan \theta$, or $-a \tan \theta$ and $y = b \sec \theta$, or $-b \sec \theta$.

Hence, (6) meets (3) in the real points $(a \tan \theta, b \sec \theta)$ and $(-a \tan \theta, -b \sec \theta)$.

III. If a pair of conjugate diameters meets the hyperbola (1) and its conjugate in P and D respectively, then

$$(i) \quad CP^2 - CD^2 = a^2 - b^2,$$

$$\text{and } (ii) \quad SP \cdot S'P = CD^2,$$

where C is the centre and S, S' the foci of (1).

If P is the point $(a \sec \theta, b \tan \theta)$, then the diameter conjugate to CP w. r. t. (1) meets the conjugate hyperbola (3) in the points $(a \tan \theta, b \sec \theta)$ and $(-a \tan \theta, -b \sec \theta)$ as shown above. Let D be the point $(a \tan \theta, b \sec \theta)$, then

$$\begin{aligned} CP^2 - CD^2 &= (a^2 \sec^2 \theta + b^2 \tan^2 \theta) - (a^2 \tan^2 \theta + b^2 \sec^2 \theta) \\ &= a^2 - b^2. \end{aligned} \quad \dots \dots (7)$$

$$\text{Again } (\S 6.14), \quad SP = ex - a = a(e \sec \theta - 1),$$

$$\text{and} \quad S'P = ex + a = a(e \sec \theta + 1);$$

$$\begin{aligned} \therefore \quad SP \cdot S'P &= a^2(e^2 \sec^2 \theta - 1) \\ &= \delta(a^2 + b^2) \sec^2 \theta - a^2 \\ &= a^2 \tan^2 \theta + b^2 \sec^2 \theta \\ &= CD^2. \end{aligned} \quad \dots \dots (8)$$

Note. If the diameter conjugate to PCP' w. r. t. (1) meets (3) in D, D' (fig. 16), then the length DD' is usually referred to as the length of the diameter conjugate to PCP' .

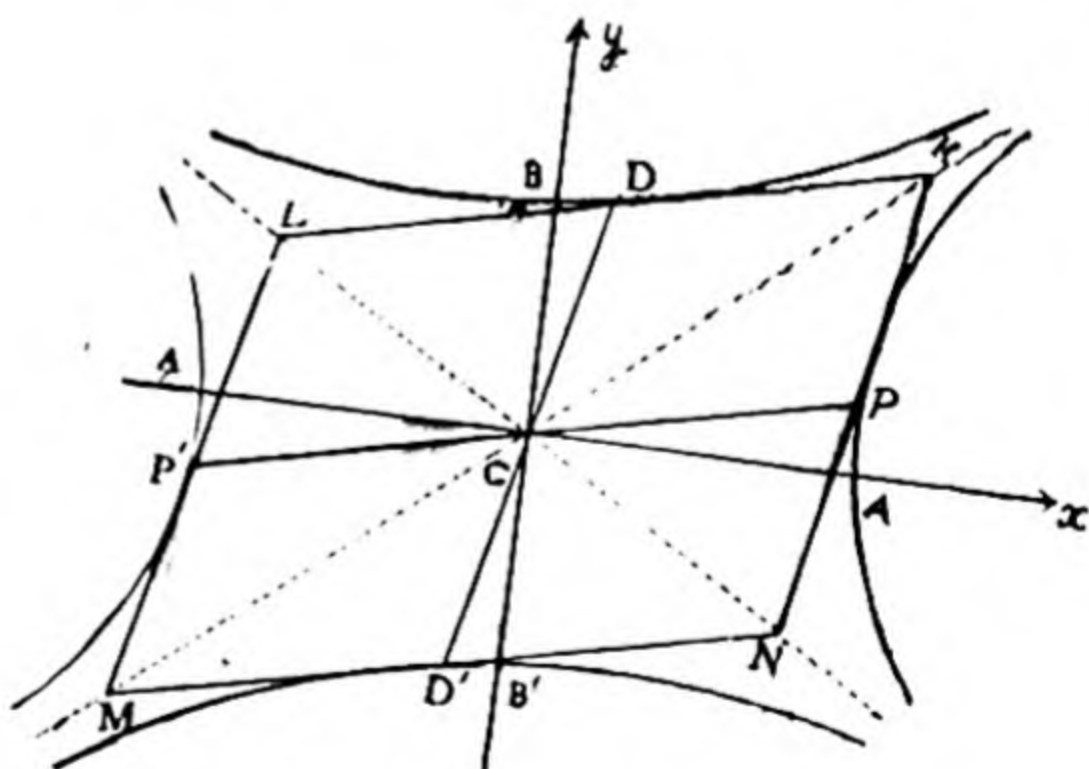


Fig. 16

It should be clearly understood, however, that PCP', DCD' cannot be looked upon as the length of conjugate diameters of

any one hyperbola. If the diameter DCD' meets the hyperbola (1) in D_1, D_1' , it can be easily shown that $CD_1^2 = -CD^2$. Thus, the relation (7) above can be written as $CP^2 + CD_1^2 = a^2 - b^2$, a result similar to the one established for the ellipse [§ 7.93, Part I].

IV. *If a pair of conjugate diameters meet the hyperbola (1) and its conjugate in P, P' and D, D' respectively, then the parallelogram formed by tangents at P, P' and D, D' is of constant area and its vertices lie on the asymptotes.*

Let the coordinates of P be $(a \sec \theta, b \tan \theta)$, then the coordinates of P', D, D' are respectively $(-a \sec \theta, -b \tan \theta)$, $(a \tan \theta, b \sec \theta)$ and $(-a \tan \theta, -b \sec \theta)$ as explained above.

$$\left. \begin{array}{l} \text{The tangent at } P \text{ is } \frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = 1, \\ \text{i. e., } \frac{x}{a} - \frac{y}{b} \sin \theta = \cos \theta, \end{array} \right\} \dots \dots (9)$$

and is thus parallel to DCD' [see (6) above].

Similarly, the tangent at P' is parallel to DCD' and the tangents at D, D' are parallel to PCP' .

Hence, the area of the parallelogram $KLMN$ (fig. 15) formed by the tangents at P, P', D, D' is equal to four times the area of the parallelogram $CPKD$.

If p be the length of the perpendicular from C on the tangent at P , then

$$\begin{aligned} p &= 1 / \sqrt{\frac{\sec^2 \theta}{a^2} + \frac{\tan^2 \theta}{b^2}}, \text{ from (9)} \\ &= \frac{ab}{\sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}}; \end{aligned}$$

\therefore area of the parallelogram $CPKD$.

$$\begin{aligned} &= CD \times p = \sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta} \times \frac{ab}{\sqrt{a^2 \tan^2 \theta + b^2 \sec^2 \theta}} \\ &= ab. \end{aligned}$$

Thus, the area of the parallelogram $KLMN$ is constant and equal to $4ab$.

Again, The tangent at $D (a \tan \theta, b \sec \theta)$ to the conjugate hyperbola is

$$\left. \begin{aligned} \frac{x}{a} \tan \theta - \frac{y}{b} \sec \theta &= -1, \\ \frac{x}{a} \sin \theta - \frac{y}{b} &= -\cos \theta, \end{aligned} \right\} \dots\dots(1)$$

$\therefore K$, the point of intersection of (9) and (10) is given by

$$\frac{x}{a} = \frac{y}{b} = \frac{\cos \theta}{1 - \sin \theta}$$

Hence, K lies on the asymptote $\frac{x}{a} - \frac{y}{b} = 0$.

Similarly, it can be shown that L, M, N also lie on the asymptotes.

Cor. Since $PK = CD = CD' = PN$, it follows that *the portion of a tangent intercepted between the asymptotes is bisected at the point of contact.*

Note. The parallelogram $KLMN$ is called the *conjugate parallelogram*.

6.7 Rectangular Hyperbola. If the asymptotes of a hyperbola be perpendicular, it is called a *rectangular hyperbola*.

The asymptotes of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ are the lines

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \text{ and are, therefore, perpendicular, if } \frac{1}{a^2} - \frac{1}{b^2} = 0,$$

i. e., if $b = a$.

Thus, the transverse and the conjugate axes of a rectangular hyperbola are equal in length and for this reason, it is also called an *equilateral hyperbola*.

The standard equation of a rectangular hyperbola is, therefore, $\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$, *i. e.*, $x^2 - y^2 = a^2$(1)

The eccentricity of the hyperbola (1) is given by

$$a^2 = a^2 (e^2 - 1), \text{ i. e., } e^2 = 2,$$

whence

$$e = \sqrt{2}.$$

Note. The various propositions for the hyperbola (1) can be derived from the corresponding propositions for the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ by putting $b = a$. For example, *the diameters*

$y=mx, y=m'x$ will be conjugate w. r. t. (1), if $mm'=1$, i. e., the diameters $y=mx, y=\frac{1}{m}x$ are always conjugate w. r. t. the hyperbola

(1) It can also be easily shown that these conjugate diameters are equally inclined to the asymptotes.

6.71. Rectangular Hyperbola referred to its Asymptotes.

The equation $x^2 - y^2 = a^2$ (1)

when referred to the asymptotes as axes takes a simpler form as shown below.

Rotating the axes through an angle -45° , the transformed equation is obtained by writing

$$\frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \text{ for } x \text{ and } \frac{-x}{\sqrt{2}} + \frac{y}{\sqrt{2}} \text{ for } y.$$

Hence, the transformed equation is

$$\frac{1}{2} (x+y)^2 - \frac{1}{2} (x-y)^2 = a^2,$$

$$i. e., \quad xy = \frac{1}{2} a^2,$$

This equation is usually written in the form

$$xy = c^2, \quad \dots\dots(2)$$

where $c^2 = \frac{1}{2} a^2$.

6.72. Propositions about the Hyperbola $xy = c^2$. As the axes are rectangular, the equations of the tangent, normal, polar etc., can be obtained by the methods of Chapter IV. Thus :

(i) The equation of the tangent at the point (x_1, y_1) on the rectangular hyperbola $xy = c^2$ (1)

$$is \quad xy_1 + x_1 y = 2c^2, \quad \dots\dots(2)$$

which can also be written as

$$\frac{x}{x_1} + \frac{y}{y_1} = 2. \quad \dots\dots(3) \quad [\because c^2 = x_1 y_1]$$

The equation (2) or (3) also represents the polar of (x_1, y_1) and the chord of contact of tangents drawn from (x_1, y_1) .

(ii) The equation of the normal at (x_1, y_1) on (1) is

$$xx_1 - yy_1 = x_1^2 - y_1^2.$$

(iii) The equation of the pair of tangents from (x_1, y_1) is

$$SS_1 = T^2,$$

where

$$S \equiv xy - c^2, \quad S_1 \equiv x_1 y_1 - c^2,$$

and

$$T \equiv \frac{1}{2} (xy_1 + x_1 y).$$

(iv) The equation of the chord with its middle point as (x_1, y_1) is

$$T = S_1,$$

where T and S_1 have the same meaning as in (iii) above.

6.73. Parametric Representation of $xy = c^2$. The coordinates of any point on the rectangular hyperbola $xy = c^2$ can be represented by $x = ct, y = c/t$, (1)

where t is a parameter, as can be seen by actual substitution.

The point $(ct, \frac{c}{t})$ is often referred to as the point 't'.

6.74. Tangent and Normal at the Point 't'

Putting $x_1 = ct$ and $y_1 = \frac{c}{t}$ in

$$\frac{x}{x_1} + \frac{y}{y_1} = 2,$$

we see that the tangent at the point $(ct, \frac{c}{t})$ on the rectangular hyperbola $xy = c^2$ is

$$\frac{x}{t} + yt = 2c, \quad \dots\dots\dots(1)$$

The normal at the point 't' is

$$y - \frac{c}{t} = t^2 (x - ct),$$

$$\text{or} \quad xt - \frac{y}{t} = c (t^2 - \frac{1}{t^2}), \quad \dots\dots\dots(2)$$

$$\text{or} \quad ct^4 - xt^3 + yt - c = 0. \quad \dots\dots\dots(3)$$

Note 1. The equation (1) can be independently obtained as follows.

The equation of the chord joining the point 't' and 't₁', on the hyperbola is

$$\begin{aligned} \frac{x - ct_1}{c(t_1 - t_2)} &= \frac{y - c/t_1}{c(\frac{1}{t_1} - \frac{1}{t_2})} \\ &= \frac{(yt_1 - c)t_2}{c(t_2 - t_1)}, \end{aligned}$$

$$\text{or} \quad x + yt_1 = c(t + t_1).$$

when $t_1 \rightarrow t$, the equation of the tangent at the point 't' is

$$x + y t^2 = 2ct,$$

i. e.,

$$\frac{x}{t} + yt = 2c.$$

Note 2. The equation (3) also shows that four normals can be drawn to $xy=c^2$ from any point.

*6.8. Equation of a Hyperbola referred to its Asymptotes.

If the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots (1)$$

be taken as the new axes, then the equation (1) is transformed into

$$XY = a \text{ constant, say } k, \quad \dots (2)$$

since $XY=0$ is now the equation of the pair of asymptotes

[§ 6. 41, Note 2].

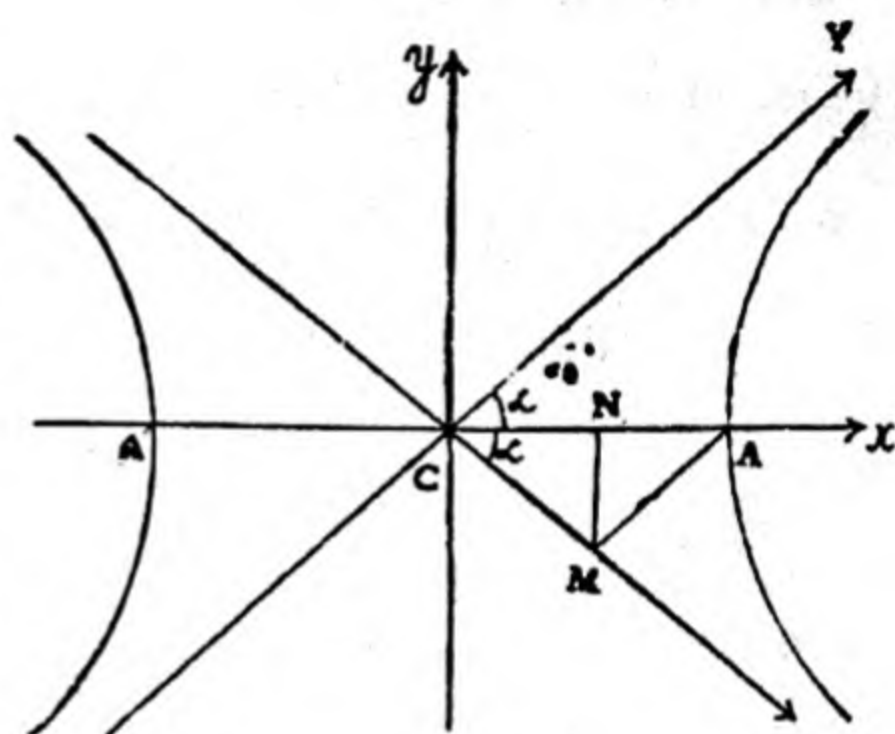


Fig. 17

Draw AM parallel to CY to meet CX in M and MN perpendicular to CA (fig.17), then

$$CM = MA = CN \sec \alpha, \text{ where } \angle ACX = \angle ACY = \alpha.$$

$$\text{Now, } \tan \alpha = \frac{b}{a} \text{ and } CN = \frac{1}{2} CA = \frac{1}{2} a;$$

$$\therefore CN \sec \alpha = \frac{1}{2} a \times \sqrt{1 + \frac{b^2}{a^2}} = \frac{1}{2} \sqrt{a^2 + b^2} = \frac{1}{2} ae.$$

Hence, the coordinates of A with reference to the new axes are $(\frac{1}{2}ae, \frac{1}{2}ae)$. Since A lies on the hyperbola (2),

$$k = \frac{1}{4} a^2 e^2 = \frac{a^2 + b^2}{4}.$$

Hence, the equation of the hyperbola (1) when referred to the asymptotes as axes is $xy = \frac{a^2 + b^2}{4}$, (3)

Cor 1. If $b=a$, the equation (3) becomes $xy = \frac{1}{2} a^2$, which is the equation of a rectangular hyperbola referred to its asymptotes as axes.

Cor.. 2. The equation of the conjugate hyperbola referred to asymptotes as axes is $xy = -\frac{a^2 + b^2}{4}$.

***6.9. Equation of a Hyperbola referred to Conjugate Diameters.**

Let the coordinates of any point on the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots\dots\dots(1)$$

referred to CA, CB as axes be (x, y) and the coordinates of the same point referred to conjugate semi-diameters CP, CD (fig. 16) as axes be (X, Y) , then the two sets of coordinates are connected by the relations

$$x = lX + mY, \quad y = l'X + m'Y,$$

where l, m, l', m' are constants depending upon the angles ACP and ACD .

The equation (1) when referred to CP, CD as axes, therefore, takes the form $AX^2 + 2HXY + BY^2 = 1$. $\dots\dots\dots(2)$

Since CP , the new X -axis bisects all chords parallel to CD , the new Y -axis, it follows that the points (X_1, Y_1) and $(X_1, -Y_1)$ both lie on (2). Thus,

$$AX_1^2 + 2HX_1Y_1 + BY_1^2 = 1 \text{ and } AX_1^2 - 2HX_1Y_1 + BY_1^2 = 1,$$

whence $H=0$.

$$\text{The equation (2) now reduces to } AX^2 + BY^2 = 1. \quad \dots\dots\dots(3)$$

If $CP = a'$ and $CD = b'$, then the point $(a', 0)$ lies on the hyperbola (1) and the point $(0, b')$ lies on the conjugate hyperbola

$$AX^2 + BY^2 = -1 \quad \therefore Aa'^2 = 1, \text{ i. e. } A = \frac{1}{a'^2},$$

$$\text{and } Bb'^2 = -1, \quad \text{i. e.,} \quad B = -\frac{1}{b'^2}.$$

Hence, the equation of a hyperbola referred to conjugate diameters as axes is of the form

$$\frac{X^2}{a'^2} - \frac{Y^2}{b'^2} = 1.$$

Ex. 1. Find the asymptotes of the hyperbola $xy - 3x - 2y = 0$ and the equation of the conjugate hyperbola. (Nagpur 1957)

Since the equation of the asymptotes differs from that of the hyperbola by a constant and the equation of the conjugate hyperbola differs from that of the asymptotes by the same constant, the equations of the asymptotes and the conjugate hyperbola are respectively

$$xy - 3x - 2y + c = 0, \quad \dots\dots\dots(1)$$

and

$$xy - 3x - 2y + 2c = 0, \quad \dots\dots\dots(2)$$

As (1) represents a pair of lines, we have

$$2. (-1). (-3/2). (\frac{1}{2}) - c (\frac{1}{2})^2 = 0, \text{ i.e., } c = 6.$$

Hence, the equation of the asymptotes is

$$xy - 3x - 2y + 6 = 0, \quad \dots\dots\dots(3)$$

and that of the conjugate hyperbola is

$$xy - 3x - 2y + 12 = 0.$$

Note. The separate equations of the two asymptotes given, by (3) are $x=2$ and $y=3$. For other methods see Exs. 1 and 2 § 7. 2,

Ex. 2. Prove that the chord, which joins the points in which a pair of conjugate diameters meets the hyperbola and its conjugate, is parallel to one asymptote and is bisected by the other. (Vikram 1960)

If any diameter of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \dots\dots\dots(1)$$

meet it in $P (a \sec \theta, b \tan \theta)$, then the conjugate diameter meets the conjugate hyperbola in $D (a \tan \theta, b \sec \theta)$ [§ 6.6, II].

Now, the equation of the chord PD is

$$\frac{y - b \sec \theta}{b (\sec \theta - \tan \theta)} = \frac{x - a \tan \theta}{a (\tan \theta - \sec \theta)},$$

or

$$\frac{x}{a} + \frac{y}{b} = \sec \theta + \tan \theta,$$

which is obviously parallel to the asymptote $\frac{x}{a} + \frac{y}{b} = 0$ of the hyperbola (1).

Again, the mid-point of PD is

$$[\frac{1}{2} a (\sec \theta + \tan \theta), \frac{1}{2} b (\sec \theta + \tan \theta)]$$

which obviously lies on the asymptote $\frac{x}{a} - \frac{y}{b} = 0$.

Hence, the result.

Ex. 3. If a circle and the rectangular hyperbola $xy=c^2$ meet in the four points ' t_r ', $r=1, 2, 3, 4$; prove that :

(i) $t_1 t_2 t_3 t_4 = 1$. (Rajputana 1950)

(ii) The centre of mean position of the four points bisects the distance between the centres of the two curves. (Delhi 1954)

(iii) The centre of the circle through the points ' t_1 ', ' t_2 ', ' t_3 ', is

$$\left[\frac{1}{2} c \left(t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3} \right), \frac{1}{2} c \left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 t_2 t_3 \right) \right]$$

(Aligarh 1950)

Let the circle $x^2 + y^2 + 2gx + 2fy + k = 0$ (1)

meet the hyperbola $xy=c^2$ in any point $(ct, \frac{c}{t})$, then

$$c^2 t^2 + \frac{c^2}{t^2} + 2gct + 2f \frac{c}{t} + k = 0, \quad [\because 't' \text{ lies on (1)}]$$

so that $c^2 t^4 + 2gct^3 + kt^2 + 2fct + c^2 = 0$,(2)

Since (2) is a biquadratic in t , it follows that the circle and the hyperbola meet in four points. If t_1, t_2, t_3, t_4 be the roots of (2), then

$$t_1 t_2 t_3 t_4 = 1, \quad \text{.....(3)}$$

$$t_1 + t_2 + t_3 + t_4 = -\frac{2g}{c}, \quad \text{.....(4)}$$

and $t_2 t_3 t_4 + t_3 t_4 t_1 + t_4 t_1 t_2 + t_1 t_2 t_3 = -\frac{2f}{c}$(5)

Dividing (5) by (3), we get

$$\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{1}{t_4} = -\frac{2f}{c} \quad \text{.....(6)}$$

Hence, the centre of the mean position of the four points is

$$\left(\frac{1}{4} c \Sigma t_i, \frac{1}{4} c \Sigma \frac{1}{t_i} \right) \text{ i. e., } \left(-\frac{g}{2}, -\frac{f}{2} \right) \text{ [from (4) and (6)],}$$

which is, therefore, the mid-point of the st. line joining $(0, 0)$ and $(-g, -c)$, i. e., the mid-point of the st. line joining the centre of the hyperbola and the centre of the circle.

Also, eliminating t_4 from (3) and (4), we have

$$-g = \frac{c}{2} \left(t_1 + t_2 + t_3 + \frac{1}{t_1 t_2 t_3} \right). \quad \text{.....(7)}$$

Similarly, $-f = \frac{c}{2} \left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1 t_2 t_3 \right)$ (8)

The equations (7) and (8) give the centre of the circle (1)

Note. It follows from (3) above that if (x_r, y_r) , $r=1, 2, 3, 4$, be the four points of intersection of the circle and the hyperbola, then

$$x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = c^4.$$

Ex. 4. If the tangent and normal to a rectangular hyperbola cut off intercepts a_1 and a_2 on one axis and b_1 and b_2 on the other, show that

$$a_1 a_2 + b_1 b_2 = 0. \quad (\text{Agra 1958})$$

The equations of the tangent and the normal at any point (x_1, y_1) on the rectangular hyperbola $xy=c^2$ are

$$\frac{x}{x_1} + \frac{y}{y_1} = 2, \quad \dots\dots(1)$$

and

$$xx_1 - yy_1 = x_1^2 - y_1^2. \quad \dots\dots(2)$$

Now, let (1) and (2) respectively cut off intercepts a_1, a_2 on the x -axis and b_1, b_2 on the y -axis, then

$$a_1 = 2x_1, \quad b_1 = 2y_1,$$

$$a_2 x_1 = x_1^2 - y_1^2, \text{ i.e. }, a_2 = \frac{x_1^2 - y_1^2}{x_1},$$

and

$$-b_2 y_1 = x_1^2 - y_1^2, \text{ i.e. } b_2 = -\frac{x_1^2 - y_1^2}{y_1}.$$

Hence,

$$a_1 a_2 + b_1 b_2 = 2x_1 \cdot \frac{x_1^2 - y_1^2}{x_1} - 2y_1 \cdot \frac{x_1^2 - y_1^2}{y_1} = 0.$$

Ex. 5. If the normals at (x_1, y_1) , (x_2, y_2) , (x_3, y_3) and (x_4, y_4) on the rectangular hyperbola $xy=c^2$ meet in a point (α, β) , prove that

$$\alpha = \Sigma x_1, \quad \beta = \Sigma y_1,$$

and

$$x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = -c^4. \quad (\text{Lucknow 1955})$$

The normal at any point 't' on $xy=c^2$ is

$$xt - \frac{y}{t} = c \left(t^2 - \frac{1}{t^2} \right).$$

If it passes through (α, β) , then

$$\alpha t - \beta/t = c \left(t^2 - 1/t^2 \right),$$

i.e.,

$$ct^4 - \alpha t^3 + \beta t - c = 0. \quad \dots\dots(1)$$

If t_1, t_2, t_3, t_4 be the four roots of (1) corresponding to the four points (x_r, y_r) , $r=1, 2, 3, 4$, then

$$\Sigma t_1 = \frac{\alpha}{c}, \text{ i.e. }, \alpha = c \Sigma t_1 = \Sigma x_1,$$

and

$$\Sigma \frac{1}{t_1} = \frac{\Sigma t_2 t_3 t_4}{t_1 t_2 t_3 t_4} = \frac{\beta}{c},$$

i. e.,
$$\beta = c \sum \frac{1}{t_1} = \sum y_1.$$

Also, since $t_1 t_2 t_3 t_4 = -1$, we have

$$x_1 x_2 x_3 x_4 = -c^4 = y_1 y_2 y_3 y_4,$$

Note. The four points (x_r, y_r) are called *co-normal points*.

Exercise 6

1. Find the equation to the hyperbola whose directrix is $2x + y = 1$, focus $(1, 1)$ and eccentricity $\sqrt{3}$. (Delhi 1947)

2. In the hyperbola $4x^2 - 9y^2 = 36$, find the axes, coordinates of the foci, the eccentricity and the latus rectum.

3. Prove that the lines $\frac{x}{a} - \frac{y}{b} = m$ and $\frac{x}{a} + \frac{y}{b} = \frac{1}{m}$ always meet on a hyperbola.

4. If the line $y = mx + \sqrt{a^2 m^2 - b^2}$ touches the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ at ' ϕ ', show that $\phi = \sin^{-1} \left(\frac{b}{am} \right)$.

5. The perpendiculars from the centre upon the tangent and normal at any point of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet them in Q and R . Find the loci of Q and R .

6. If the chord through the points ' θ ' and ' ϕ ' on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through a focus, prove that

$$\tan \frac{\theta}{2} \tan \frac{\phi}{2} + \frac{e-1}{e+1} = 0.$$

7. The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meets the axes in M and N , and the lines MP and NP are drawn at right angles to the axes; prove that the locus of P is the hyperbola $a^2 x^2 - b^2 y^2 = (a^2 + b^2)^2$. (Punjab 1954)

8. Prove that the locus of the poles of normal chords w. r. t. the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is the curve

$$y^2 a^6 - x^2 b^6 = (a^2 + b^2)^2 x^2 y^2.$$

9. From points on the circle $x^2 + y^2 = a^2$ tangents are drawn

to the hyperbola $x^2 - y^2 = a^2$; prove that the locus of the middle points of the chords of contact is the curve

$$(x^2 - y^2)^2 = a^2 (x^2 + y^2). \quad (\text{Agra 1958 S})$$

10. Show that the locus of poles w. r. t. the parabola $y^2 = 4ax$ of tangents to the hyperbola $x^2 - y^2 = a^2$ is the ellipse $4x^2 + y^2 = 4a^2$. (*Agra '55 S; Aligarh '58*)

11. Prove that the polar of any point on $\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$ w. t. r. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ touches $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. (*Delhi 1956*)

12. Prove that, if the normals at the four points $(a \sec \theta_r, b \tan \theta_r)$, $r = 1, 2, 3, 4$, on hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet in a point, then will $\theta_1 + \theta_2 + \theta_3 + \theta_4 = (2n + 1)\pi$, n being an integer,

and $\sin(\theta_1 + \theta_2) + \sin(\theta_2 + \theta_3) + \sin(\theta_3 + \theta_1) = 0$.

13. Prove that the product of the sum of the abscissae of four co-normal points of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and the sum of their reciprocals equals four. (*Delhi Hon's '56*)

14. Find the equation of the hyperbola, whose asymptotes are the straight lines $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$, and which passes through the point $(1, -1)$.

Find also the equation of the conjugate hyperbola.

(*Rajputana 1953*)

15. Prove that the asymptotes of $xy = hx + ky$ are $x = k$ and $y = h$. (*Agra 1955*)

16. From any point of one hyperbola tangents are drawn to another which has the same asymptotes; show that the chord of contact cuts off a constant area from the asymptotes.

(*Allahabad 1950*)

17. Prove that the polar of any point on an asymptote of a hyperbola w. r. t. the hyperbola is parallel to that asymptote.

(*Delhi 1957*)

18. If e and e' be the eccentricities of a hyperbola and its conjugate, prove that

$$\frac{1}{e^2} + \frac{1}{e'^2} = 1 \quad (\text{Delhi '58; Osmania '60})$$

19. If two fixed points on a hyperbola are joined to any variable point on the curve, show that the portion, which these lines intercept on either asymptote, is constant.

(*Allahabad 1957*)

20. Find the asymptotes of the hyperbola given by

$$x = a \tan (\theta + \alpha), \quad y = b \tan (\theta + \beta),$$

where θ is a parameter.

(*Allahabad 1957*)

21. Show that the circle described on the straight line joining the foci of a hyperbola as diameter passes through the foci of its conjugate hyperbola.

22. Show that the tangents at the extremities of the latera recta of the hyperbola pass through the vertices of the conjugate hyperbola.

23. Show that the polar of any point on the conjugate hyperbola w. r. t. the given hyperbola touches the conjugate hyperbola.

24. A straight line cuts off a triangle of constant area from two given perpendicular lines; show that the locus of the centroid of the triangle is a rectangular hyperbola whose asymptotes are the given lines.

(*Punjab 1953*)

25. Prove that chords of a hyperbola, which touch the conjugate hyperbola, are bisected at the point of contact.

26. From any point of one hyperbola tangents are drawn to another which has the same asymptotes; show that the chord of contact cuts off a constant area from the asymptotes.

(*Allahabad 1950*)

27. Find the coordinates of the foci and the equation of the directrices of the rectangular hyperbola, $xy = c^2$.

(*Travancore '47*)

28. Prove that if a chord of a rectangular hyperbola subtends a right angle at a vertex, it is parallel to the transverse axis.

29. A series of rectangular hyperbolas have the same asymptotes; show that if two lines form a pair of conjugate diameters w. r. t. one of them, they are so w. r. t. each of them.

30. Prove that $y - mx = 0$ and $y + mx = 0$ are conjugate diameters of $xy = c^2$ for all values of m . (U. P. S. C. '52)

31. Prove that the rectangle contained by the intercepts made by any tangent to a hyperbola on its asymptotes is constant. (Agra 1956)

32. The two lines $x - a = 0, y - \beta = 0$ are conjugate w. r. t. the hyperbola $xy = c^2$. Prove that (α, β) lies on the hyperbola $xy = 2c^2$. (Allahabad '38)

33. Show that the normal to the rectangular hyperbola $xy = c^2$ at the point ' t ' meets the curve again at the point ' t' ', such that $t^3 t' = -1$. (Agra 1957, '59)

34. A circle cuts the rectangular hyperbola $xy = 1$ in the points $(x_r, y_r), r = 1, 2, 3, 4$. Prove that

$$x_1 x_2 x_3 x_4 = y_1 y_2 y_3 y_4 = 1. \quad (\text{Rajputana 1954})$$

35. A rectangular hyperbola whose centre is C is cut by any circle of radius r in the four points P, Q, R, S ; prove that

$$CP^2 + CQ^2 + CR^2 + CS^2 = 4r^2,$$

(Rajputana '58; Allahabad '60)

36. Q is any point on a rectangular hyperbola and PP' is any diameter. Prove that the bisectors of the angle PQP' are parallel to the asymptotes. (Gujarat 1957)

37. Prove that the locus of the poles of normal chords of the rectangular hyperbola $xy = c^2$ is the curve

$$(x^2 - y^2)^2 + 4c^2 xy = 0.$$

(Rajputana '56; Aligarh '59)

38. If a hyperbola be rectangular and its equation be $xy = c^2$, prove that the locus of the middle points of chords of constant length $2d$ is $(x^2 + y^2)(xy - c^2) = d^2 xy$.

(Banaras; '49; Allahabad '56)

39. The normals at the three points P, Q, R , on a rectangular hyperbola intersect at a point on the curve. Prove that the centre of the hyperbola is the centroid of the triangle PQR .

(Poona 1953)

40. Prove that if the normals at P, Q, R, S on a rectangular hyperbola intersect in a point, then the circle PQR passes through the other extremity of the diameter through S .

(Allahabad '55)

41. Show that the locus of the middle points of normal chords of the rectangular hyperbola $x^2 - y^2 = a^2$ is

$$(y^2 - x^2)^3 = 4a^2 x^2 y^2.$$

(Agra '56; Allahabad '58)

42. If a triangle is inscribed in a rectangular hyperbola, prove that the orthocentre of the triangle lies on the curve.

(Mysore '59; Agra '60)

43. Show that the straight line $y = mx + 2c\sqrt{-m}$ always touches the hyperbola $xy = c^2$, and that the point of contact is

$$\left(\frac{c}{\sqrt{-m}}, c\sqrt{-m} \right).$$

44. If four points be taken on a rectangular hyperbola such that the chord joining any two is perpendicular to the chord joining the other two, and if $\alpha, \beta, \gamma, \delta$ be the inclinations to either asymptote of the straight lines joining these points respectively to the centre, prove that

$$\tan \alpha \tan \beta \tan \gamma \tan \delta = 1.$$

45. Prove that the tangent at any point of a hyperbola cuts off a triangle of constant area from the asymptotes.

[**Hint** :—Take the asymptotes as axes, or apply § 6.6, IV]

46. If a line cuts the same branch of a hyperbola in P and Q and the asymptotes in P' and Q' , show that $PP' = QQ'$.

(Osmania 1957)

47. The four normals at the points of a rectangular hyperbola $xy = c^2$ in which it is met by the chords

$x \cos \alpha + y \sin \alpha - p = 0$ and $p(x \sin \alpha - y \cos \alpha) - c^2 \cos 2\alpha = 0$ are concurrent.

Answers

1. $7x^2 + 12xy - 2y^2 - 2x + 4y - 7 = 0.$

2. $6, 4; (\pm\sqrt{13}, 0); \frac{1}{2}\sqrt{13}; 2\frac{2}{3}.$

3. Locus of intersection is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
5. $(x^2 + y^2)^2 = a^2x^2 - b^2y^2$; $(x^2 + y^2)^2 (a^2y^2 - b^2x^2) = (a^2 + b^2)^2 x^2 y^2$.
14. $3x^2 + 10xy + 8y^2 + 14x + 22y + 7 = 0$:
 $3x^2 + 10xy + 8y^2 + 14x + 22y + 23 = 0$.
20. $x + a \cot (a - \beta) = 0$, $y - b \cot (a - \beta) = 0$.
27. $(c\sqrt{2}, c\sqrt{2})$, $(-c\sqrt{2}, -c\sqrt{2})$; $x + y = \pm c\sqrt{2}$.

Miscellaneous Exercise

(Parabola, Ellipse and Hyperbola)

1. A straight line touches both $x^2 + y^2 = 2a^2$ and $y^2 = 8ax$. Show that its equation is $y = \pm(x + 2a)$. (Calcutta 1955)

2. Tangents to the parabola $y^2 = 4ax$ are drawn at points whose abscissae are in the ratio $m^2 : 1$. Prove that the locus of their points of intersection is the curve $y^2 = (m^{\frac{1}{2}} + m^{\frac{1}{2}})^2 ax$. (Osmania '60)

3. If a tangent to the parabola $y^2 = 4ax$ meets the parabola $y^2 = -4ax$ at Q and R , show that the tangents at Q and R to the second parabola intersect on the first.

4. Prove that the length of the chord of contact of the tangents drawn from the point (x_1, y_1) to the parabola $y^2 = 4ax$ is

$$\frac{1}{a} \sqrt{[(y_1^2 + 4a^2)(y_1^2 - 4ax)]}. \quad (\text{Punjab 1947})$$

5. The perpendicular from a point P upon its polar w. r. t. the parabola $y^2 = 4ax$ is of constant length k . Find the locus of the point P . (Calcutta 1938)

6. Show that the locus of the poles of chords of the parabola $y^2 = 4ax$ which subtend a constant angle α at the vertex is

$$4(y^2 - 4ax) = (x + 4a)^2 \tan^2 \alpha. \quad (\text{Aligarh 1958})$$

7. If two of the normals to a parabola intersect at right angles, prove that the chord joining their feet passes through the focus.

8. Tangents are drawn from a variable point P to the parabola $y^2 = 4ax$ such that they form a triangle of constant area c^2 with the tangent at the vertex. Show that the locus of P is

$$x^2 (y^2 - 4ax) = 4c^4. \quad (\text{Travancore 1949})$$

9. Prove that the locus of the middle points of all tangents drawn from points on the directrix to the parabola $y^2 = 4ax$ is

$$y^2 (2x + a) = a (3x + a)^2. \quad (\text{Agra 1959})$$

10. Show that the locus of the point of intersection of tangents to $y^2=4ax$ which intercept a constant length d on the directrix is $(y^2-4ax)(x+a)^2=d^2x^2$. (Mysore 1952)

11. Show that the line $lx+my=n$ will cut the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points whose eccentric angles differ by $\frac{\pi}{2}$, if $a^2l^2+b^2m^2=2n^2$. (Aligarh 1958)

12. If two concentric ellipses be such that the foci of one lie on the other, and if e, e' be their eccentricities, show that their axes are inclined at an angle

$$\cos^{-1} \frac{\sqrt{(e^2+e'^2-1)}}{ee'} \quad (\text{Allahabad 1955})$$

13. Show that the locus of the point of intersection of the tangents to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose chord of contact subtends a right angle at the centre is the concentric ellipse

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2}+\frac{1}{b^2}.$$

14. Tangents are drawn from any point on the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$ to the conic $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$; prove that the normals at the points of contact meet on the conic

$$a^2x^2+b^2y^2=\frac{1}{4}(a^2-b^2)^2. \quad (\text{Utkal 1946})$$

15. Show that the locus of the pole, w. r. t. the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, of any tangent to the circle $x^2+y^2=c^2$ is

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{c^2}.$$

16. Find the locus of the middle points of chords of contact of tangents drawn to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ from points on the director circle. (Rajputana '57; Vikram '60)

17. Show that the locus of the middle points of normal chords of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is the curve

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2 \left(\frac{a^6}{x^2}+\frac{b^6}{y^2}\right)=(a^2-b^2)^2$$

(Rajputana 1958)

18. Show that the locus of the middle points of chords, of constant length $2c$, of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) \left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right) + \frac{c^2}{a^2 b^2} \left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = 0.$$

(Aligarh 1954)

19. A parallelogram circumscribes the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and two of its angular points are on the lines $x^2 - h^2 = 0$; prove that the other two are on the conic

$$\frac{x^2}{a^2} + y^2 (1 - a^2/h^2)/b^2 - 1 = 0. \quad (\text{Agra 1960})$$

20. Prove that the straight lines joining the centre of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the intersections of the straight line $y = mx + \sqrt{\frac{1}{2}(a^2 m^2 + b^2)}$ with the ellipse are conjugate diameters. (Vikram 1959)

21. If PCP' and DCD' are a pair of conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and Q is any point on the circle $x^2 + y^2 = c^2$, show that

$$PQ^2 + DQ^2 + P'Q^2 + D'Q^2 = 2(a^2 + b^2 + 2c^2).$$

(Lucknow 1941)

22. Show that the feet of the normals from a point (h, k) to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ lie on a rectangular hyperbola which passes through $(0,0)$ and (h, k) . (Agra 1948)

Note. The locus is known as *Apollonian Hyperbola*.

23. A series of hyperbolas is drawn having a common transverse axis of length $2a$. Prove that the locus of a point P on each hyperbola, such that its distance from the transverse axis is equal to its distance from an asymptote, is the curve

$$(x^2 - y^2)^2 = 4x^2 (x^2 - a^2). \quad (\text{Allahabad 1942})$$

24. A straight line is drawn parallel to the conjugate axis of a hyperbola to meet it and the conjugate hyperbola in the

points P and Q ; show that the tangents at P and Q meet on the curve

$$\frac{y^4}{b^4} \left(\frac{y^2}{b^2} - \frac{x^2}{a^2} \right) = -\frac{4x^2}{a^2},$$

and that the normals meet on the x -axis.

25. Show how to find the coordinates of the vertices of a triangle inscribed in the hyperbola $xy=c^2$, the sides of the triangle being parallel to the lines $y+lx=0$, $y+mx=0$, $y+nx=0$. Show that if l, m, n , vary, the area of the triangle is always proportional to

$$\frac{(m-n)(n-l)(l-m)}{lmn}.$$

26. If $x+iy=\sqrt{(\phi+i\psi)}$, where ϕ and ψ are real parameters, prove that $\phi=\text{const.}$ and $\psi=\text{const.}$ represent two systems of rectangular hyperbolas cutting at right angles.

(Delhi Hon's 1954)

27. At any point P on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ a tangent is drawn to meet the asymptotes in Q and R . If C be the centre of the hyperbola, show that the locus of the centre of the circle circumscribing the triangle CQR is

$$4(a^2x^2 - b^2y^2) = (a^2 + b^2)^2. \quad (\text{Agra 1958})$$

28. Prove that the feet of the normals from a point (h, k) to the hyperbola lie on the curve

$$(a^2 + b^2)xy + b^2kx - a^2hy = 0.$$

[Note. This curve is known as *Apollonian Hyperbola*.]

Answers

5. $(y^2 - 4ax)^2 = k^2 (y^2 + 4a^2).$

16. $x^2 + y^2 = (a^2 + b^2) \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} \right)^2.$

22. $(a^2 - b^2)xy + b^2kx - a^2hy = 0.$

CHAPTER VII

THE TRACING OF CONICS

7.1. General Equation of the Second Degree represents a Conic.

In the preceding chapters we have seen that the equation of a conic is always of the second degree. We shall now show that *the general equation of the second degree*.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots\dots(1)$$

always represents a conic.

To remove the xy -term in (1), rotate the axes through an angle θ given by $\tan 2\theta = \frac{2h}{a-b}$ (§ 3.12) so that the transformed equation is

$$a'X^2 + b'Y^2 + 2g'X + 2f'Y + c' = 0, \quad \dots\dots(2)$$

where $a+b=a'+b'$, and $ab-h^2=a'b'$,
by the theory of invariants (§ 3.13).

The following cases arise now :

Case I. If $ab-h^2 \equiv a'b' = 0$, then either $a'=0$, or $b'=0$. Suppose that $a'=0$, then equation (2) can be written as

$$b'(Y + f'/b)^2 = -2g'X - c' + f'^2/b, \quad \dots\dots(3)$$

which represents a parabola with its axis parallel to the X -axis, provided $g' \neq 0$. If $g'=0$, then the condition that (2) may represent a pair of straight lines is satisfied (§ 1.5).

Similarly, if $b'=0$ and $f' \neq 0$, equation (2) represents a parabola with its axis parallel to the Y -axis.

Case II. If $ab-h^2 \equiv a'b' \neq 0$, then neither a' nor b' is zero and equation (2) can be written as

$$a' \left(X + \frac{g'}{a'} \right) + b' \left(Y + \frac{f'}{b'} \right)^2 = \frac{g'^2}{a'} + \frac{f'^2}{b'} - c' = k', \text{ say} \quad \dots\dots(4)$$

which represents an *ellipse* or a *hyperbola* with centre at $\left(-\frac{g'}{a'}, -\frac{f'}{b'}\right)$ according as the signs of a' and b' are like or unlike, provided $k' \neq 0$. If $k' = 0$, then (4) represents a pair of *straight lines* (real or imaginary).

When a' and b' have like signs, then $a'b'$, i. e., $ab - h^2$ is positive and when a' and b' have unlike signs, then $a'b'$ i. e., $ab - h^2$ is negative.

When a' and b' have unlike signs and in addition $a' + b' = 0$, i. e., $a + b = 0$, then (4) or (1) represents a *rectangular hyperbola*.

Thus, equation (1) always represents one of the conics (see § 6.1, part I).

For another method of investigation of the nature of the conic (1), see § 7.2, Cor. 2.

Note 1. If $\frac{a'}{k'}$ and $\frac{b'}{k'}$ are both negative, then no real values of X and Y satisfy (4). Because of the form of equation (4), however, the curve is called an *imaginary ellipse*.

Note 2. If $a = b$ and $h = 0$, then $ab - h^2$ is positive. In this case equation (1) represents a circle [see § 4.11, Part I]. In § 7.16, Part I it has been shown that a circle is a limiting case of an ellipse.

Note 3. In the axes are oblique, we can change to rectangular axes with the same origin and proceed as above.

7.11. Summary. In investigating the nature of the conic represented by the general equation.

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots\dots(1)$$

the various results obtained so far can be summed up conveniently as shown below. In what follows, Δ stands for the expression $abc + 2fgh - af^2 - bg^2 - ch^2$, i. e.,

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Condition satisfied	Nature of the Conic
1. $\Delta=0, ab-h^2 \neq 0$.	A pair of st. lines, real or imaginary.
2. $\Delta=0, ab-h^2=0$.	A pair of parallel st. lines.
3. $\Delta \neq 0, ab-h^2=0$.	A parabola.
4. $\Delta \neq 0, ab-h^2 > 0$.	An ellipse
5. $\Delta \neq 0, a=b, h=0$.	A circle
6. $\Delta \neq 0, ab-h^2 < 0$.	A hyperbola
7. $\Delta \neq 0, ab-h^2 < 0,$ $a+b=0$.	A rectangular hyperbola

Note. The expression denoted by Δ is called the *discriminant* of the equation (1), or of the expression on the L. H. S. of (1).

7.2. Asymptotes. To find the equation of the asymptotes of the conic represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots\dots(1)$$

We have seen (§ 6.5) that the equation of the asymptotes differs from that of the conic only by a constant.

Hence, the asymptotes of (1) are given by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c + c' = 0, \quad \dots\dots(2)$$

Where c' is so chosen that (2) represents a pair of straight lines.

The condition that (2) may represent a pair of straight lines is

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c+c' \end{vmatrix} = 0,$$

whence

$$c' (ab-h^2) + \Delta = 0,$$

i. e.,

$$c' = -\frac{\Delta}{ab-h^2}, \text{ where}$$

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

The equation of the asymptotes of (1) is therefore

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{\Delta}{ab - h^2} = 0. \quad \dots\dots(3)$$

Cor. 1. The straight lines given by

$$ax^2 + 2hxy + by^2 = 0 \quad \dots\dots(4)$$

are parallel to the asymptotes of the conic (1) and are real and different, real and coincident or imaginary according as

$$ab - h^2 < 0, = 0 \text{ or } > 0.$$

Now, the asymptotes of a hyperbola are real and different, those of a parabola are coincident and lie at infinity, while those of an ellipse are imaginary (§ 6.4).

Hence, the conic (1) is a hyperbola, a parabola or an ellipse according as $ab - h^2 < 0, = 0$ or > 0 .

If $ab - h^2 < 0$ and further if $a + b = 0$, then the asymptotes (3) are at right angles and the hyperbola (1) is rectangular.

Cor 2. When (1) represents a hyperbola, the equation of the conjugate hyperbola is

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c - \frac{2\Delta}{ab - h^2} = 0,$$

since the equation of the conjugate hyperbola differs as much from that of the asymptotes as does the equation of the asymptotes from that of the hyperbola.

Ex 1. If (\bar{x}, \bar{y}) be the centre of the hyperbola

$$f(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

prove that the equation of the asymptotes is

$$f(x, y) = f(\bar{x}, \bar{y}).$$

(Agra '41; Banaras '55)

Method I. If (\bar{x}, \bar{y}) be the centre, then (§ 4.7)

$$a\bar{x} + h\bar{y} + g = 0, \quad h\bar{x} + b\bar{y} + f = 0,$$

so that $\bar{x}(a\bar{x} + h\bar{y} + g) + \bar{y}(h\bar{x} + b\bar{y} + f) = 0$,

i. e., $a\bar{x}^2 + 2h\bar{x}\bar{y} + b\bar{y}^2 + g\bar{x} + f\bar{y} = 0$;

$\therefore f(\bar{x}, \bar{y}) \equiv g\bar{x} + f\bar{y} + c = k$, say. \dots\dots(1)

Thus, we have the equations

$$a\bar{x} + h\bar{y} + g = 0,$$

$$h\bar{x} + b\bar{y} + f = 0,$$

and

$$g\bar{x} + f\bar{y} + (c - k) = 0.$$

Eliminating \bar{x} , \bar{y} from these equations, we get

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c-k \end{vmatrix} = 0, \text{ i. e., } \Delta - k(ab - h^2) = 0;$$

$$\therefore k = \frac{\Delta}{ab - h^2} = f(\bar{x}, \bar{y}), \text{ from (1).}$$

Hence, the equation of the asymptotes is (§ 7.2)

$$\begin{aligned} f(x, y) &= -\frac{\Delta}{ab - h^2} \\ &= f(\bar{x}, \bar{y}). \end{aligned}$$

Method II. Since the equation of the asymptotes differs from that of the hyperbola by a constant (§ 6.5) and since the asymptotes are the tangents drawn from the centre of the hyperbola (§ 6.4), it follows that the required equation is of the form

$$f(x, y) = \lambda, \quad \dots \dots (2)$$

Where the constant λ is such that (\bar{x}, \bar{y}) lies on (2);

$$\therefore f(\bar{x}, \bar{y}) = \lambda.$$

Hence, the equation of the asymptotes is $f(x, y) = f(\bar{x}, \bar{y})$.

Note. From (1), it follows that the equation of the asymptotes can also be written as

$$f(\bar{x}, \bar{y}) = g\bar{x} + f\bar{y} + c,$$

$$\text{or } ax^2 + 2hxy + by^2 + 2gx + 2fy - (g\bar{x} + f\bar{y}) = 0.$$

Ex. 2.. Find the equation of the asymptotes of the conic

$$3x^2 - 2xy - 5y^2 + 7x - 9y = 0. \quad (\text{Agra 1943})$$

$$\textbf{Method I.} \text{ Here } \Delta = \begin{vmatrix} 3 & -1 & 7/2 \\ -1 & -5 & -9/2 \\ 7/2 & -9/2 & 0 \end{vmatrix} = 32 \text{ and } ab - h^2 = -16;$$

\therefore the equation of the asymptotes is (§ 7.2.)

$$3x^2 - 2xy - 5y^2 + 7x - 9y - \frac{\Delta}{ab - h^2} = 0,$$

$$\text{i. e. } 3x^2 - 2xy - 5y^2 + 7x - 9y + 2 = 0.$$

Method II. The centre of the conic is given by the equations (§ 4.7)

$$3x - y + 7/2 = 0, \text{ and } x + 5y + 9 = 0;$$

$$\therefore \text{ the centre is the point } \left(-\frac{11}{8}, -\frac{5}{8} \right).$$

Hence, by Ex. 1 above, the equation of the asymptotes is

$$3x^2 - 2xy + 5y^2 + 7x - 9y + k = 0, \quad \dots\dots(1)$$

where k is such that the centre lies on (1); $\therefore k = 2$.

7.3. To find the equation of the axis and the length of the latus rectum of the parabola represented by

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots\dots(1)$$

If (1) represents a parabola, then $ab - h^2 = 0$, i. e., the second degree terms form a perfect square. Hence, equation (1) can be written as

$$a^2x^2 + 2ahxy + aby^2 + 2agx + 2afy + ac = 0,$$

$$\text{or} \quad (ax + hy)^2 + 2agx + 2afy + ac = 0, \quad \dots\dots(2)$$

$$\text{since} \quad ab = h^2.$$

As a parabola is the locus of a point the square of whose distance from a straight line (axis) bears a constant ratio (length of the latus rectum) to its distance from a perpendicular straight line (tangent at the vertex), we write (2) in a form which expresses this property [see § 6.11, Part I].

Now, writing (2) as

$$(ax + hy + \lambda)^2 = 2a(\lambda - g)x + 2(h\lambda - af)y + \lambda^2 - ac, \quad \dots\dots(3)$$

we choose λ such that the lines

$$ax + hy + \lambda = 0, \quad \dots\dots(4)$$

$$\text{and} \quad 2a(\lambda - g)x + 2(h\lambda - af)y + \lambda^2 - ac = 0 \quad \dots\dots(5)$$

are perpendicular to each other,

$$\text{i.e., such that} \quad a^2(\lambda - g) + h(h\lambda - af) = 0,$$

$$\text{whence} \quad \lambda = \frac{a(ag + hf)}{a^2 + h^2} = \frac{ag + hf}{a + b}. \quad \dots\dots(6) \quad [\because h^2 = ab]$$

For this value of λ , the equations (4) and (5) respectively represent the axis of the parabola and the tangent at the vertex. The vertex of the parabola is the point of intersection of (4) and (5).

To find the latus rectum of (1), we write (3) as

$$\begin{aligned}
 & \left(\frac{ax + hy + \lambda}{\sqrt{a^2 + h^2}} \right) (a^2 + h^2) \\
 &= \pm 2 \sqrt{a^2 (\lambda - g)^2 + (h\lambda - af)^2} \times \\
 & \left[\frac{2a(\lambda - g)x + 2(h\lambda - af)y + \lambda^2 - ac}{\pm 2\sqrt{a^2 (\lambda - g)^2 + (h\lambda - af)^2}} \right], \\
 \text{i. e.,} \quad & \left(\frac{ax + hy + \lambda}{\sqrt{a^2 + ab}} \right)^2 \\
 &= \pm \frac{2\sqrt{\{a^2 (\lambda - g)^2 + (h\lambda - af)^2\}}}{a(a+b)} \times \\
 & \left[\frac{2a(\lambda - g)x + 2(h\lambda - af)y + \lambda^2 - ac}{\pm 2\sqrt{a^2 (\lambda - g)^2 + (h\lambda - af)^2}} \right], \dots\dots(7)
 \end{aligned}$$

which shows that the length of the latus rectum is the numerical value of $\frac{2}{a(a+b)} [a^2 (\lambda - g)^2 + (h\lambda - af)^2]^{\frac{1}{2}}$ which is

$$\begin{aligned}
 &= \frac{2}{a(a+b)} \left[a^2 \left(\frac{ag + hf}{a+b} - g \right)^2 + \left(h \frac{ag + hf}{a+b} - af \right)^2 \right]^{\frac{1}{2}}, \text{ from (6)} \\
 &= \frac{2}{a(a+b)^2} [a^2 (hf - bg)^2 + a^2 (gh - af)^2]^{\frac{1}{2}} \quad (\because h^2 = ab) \\
 &= \frac{2}{(a+b)^2} [b(af^2 + bg^2 - 2fgh) + a(bg^2 + af^2 - 2fgh)]^{\frac{1}{2}} \\
 & \quad (\because h^2 = ab) \\
 &= \frac{2\Delta^{\frac{1}{2}}}{(a+b)^{\frac{3}{2}}}, \dots\dots(8)
 \end{aligned}$$

where Δ , the discriminant of (1) $= af^2 + bg^2 - 2fgh$.

Note 1. In order to determine the correct sign to be prefixed to the expression for the perpendicular in (7), we have to find on which side of the tangent of the vertex the parabola lies. This can be decided with the help of the intersections of (1) with the axes. See Ex., § 7.31.

Note 2. The equation (1) when referred to the axis of the parabola as x -axis and the tangent at its vertex as y -axis

becomes
$$y^2 = \pm \frac{2\Delta^{\frac{1}{2}}}{(a+b)^{\frac{3}{2}}} x.$$

Note 3. Substituting the value of λ in (5), we get

$$2a \left(\frac{ag+hf}{a+b} - g \right) x + 2 \left(h \frac{ag+hf}{a+b} - af \right) y + \left(\frac{ag+hf}{a+b} \right)^2 - c = 0,$$

$$\text{or} \quad (hf-bg)x + (gh-af)y + k = 0, \quad \dots\dots(9)$$

$$\text{where} \quad k = \frac{(ag+hf)^2 - c(a+b)^2}{2a(a+b)}.$$

Since $hf-bg = hf - \frac{h^2}{a}g = \frac{h}{a}(af-gh)$, equation, (9) becomes

$$hx - ay + k' = 0, \quad \dots\dots(10)$$

$$\text{where} \quad k' = \frac{ak}{af-gh} = \frac{(ag+hf)^2 - c(a+b)^2}{2(a+b)(af-gh)}.$$

Thus, equation (10) represents the tangent at the vertex of the parabola (1).

7.31. Focus of the Parabola

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad \dots\dots(1)$$

The vertex (x_0, y_0) of the parabola (1) is the point of intersection of the lines (§ 7.3)

$$ax + hy + \lambda = 0 \quad \text{and} \quad hx - ay + k' = 0,$$

$$\text{where} \quad \lambda = \frac{ag+hf}{a+b} \quad \text{and} \quad k' = \frac{(ag+hf)^2 - c(a+b)^2}{2(a+b)(af-gh)}.$$

If the axis of the parabola makes an angle θ with the x -axis, then $\tan \theta = -\frac{a}{h}$, i. e.; $\cos \theta = \pm \frac{h}{\sqrt{a^2+h^2}}$ and

$$\sin \theta = \mp \frac{a}{\sqrt{a^2+h^2}}.$$

$$\text{Now, the length of the latus rectum} = \frac{2\Delta^{\frac{1}{2}}}{(a+b)^{\frac{3}{2}}} = 4p, \text{ say;}$$

\therefore the coordinates of the focus are given by

$$\frac{x-x_0}{\cos \theta} = \frac{y-y_0}{\sin \theta} = p.$$

Hence, the focus is the point $(x_0 + p \cos \theta, y_0 + p \sin \theta)$ where θ and p are as given above.

Note 1. The correct sign of the value of $\cos \theta$ and $\sin \theta$ in the discussion above is determined as in § 7.3, Note 1. See Ex. below.

Note 2. If the coordinates of the focus be (x_1, y_1) , the equation of the latus rectum is

$$hx - ay = hx_1 - ay_1, \quad \dots\dots(2)$$

since it is parallel to the tangent at the vertex.

Note 3. The coordinates (x_1, y_1) of the focus can be found as follows also.

Let the directrix, which is parallel to the tangent at the vertex, be represented by the equation $hx - ay + \lambda = 0$, then the equation of the parabola is

$$(x - x_1)^2 + (y - y_1)^2 = \frac{(hx - ay + \lambda)^2}{h^2 + a^2}. \quad \dots\dots(3)$$

Comparing Coefficients in (1) and (3), we can find λ , x_1 , and y_1 . This method, however, is not convenient and is mostly of theoretical interest.

Ex. Trace the curve

$$9x^2 + 24xy - 16y^2 - 2x + 14y + 1 = 0,$$

and find the coordinates of its focus and equation of its directrix.

(Agra 1956, '60)

$$\text{Here } \Delta = \begin{vmatrix} 9 & 12 & -1 \\ 12 & 16 & 7 \\ -1 & 7 & 1 \end{vmatrix} = 9(16 - 49) - 12(12 + 7) - (84 + 16) \neq 0;$$

and $ab - h^2 = 0$; \therefore the curve is a parabola.

The given equation can be written as

$$(3x + 4y + \lambda)^2 = 2(1 + 3\lambda)x + 2(4\lambda - 7)y + \lambda^2 - 1. \quad \dots\dots(1)$$

If the lines $3x + 4y + \lambda = 0$,

and $2(1 + 3\lambda)x + 2(4\lambda - 7)y + \lambda^2 - 1 = 0$
are at right angles, then

$$3(1 + 3\lambda) + 4(4\lambda - 7) = 0,$$

$$\text{whence } \lambda = \frac{28 - 3}{9 + 16} = 1.$$

The equation (1) now becomes

$$\begin{aligned} (3x + 4y + 1)^2 &= 8x - 6y \\ &= 2(4x - 3y), \end{aligned}$$

which can be written as

$$\left(\frac{3x + 4y + 1}{5} \right)^2 = \pm \frac{2}{5} \cdot \frac{4x - 3y}{5}. \quad \dots\dots(2)$$

Thus, the equation of the axis of the parabola is

$$3x + 4y + 1 = 0, \quad \dots\dots(3)$$

and that of the tangent at the vertex is

$$4x - 3y = 0. \quad \dots\dots(4)$$

Solving (3) and (4), we find that the coordinates of the vertex A are $\left(-\frac{3}{25}, -\frac{4}{25}\right)$.

Also, the latus rectum of the parabola is $\frac{2}{5}$.

To determine the correct sign in (2), we have to find on which side of the tangent at the vertex the parabola lies. The parabola meets the x -axis in points given by

$$9x^2 - 2x + 1 = 0,$$

which shows that both the intersections are imaginary. The parabola meets the y -axis in points for which

$$16y^2 + 14y + 1 = 0,$$

$$\text{i.e. } y = \frac{-7 \pm \sqrt{33}}{16} = -0.08 \text{ and } -0.8 \text{ approximately.}$$

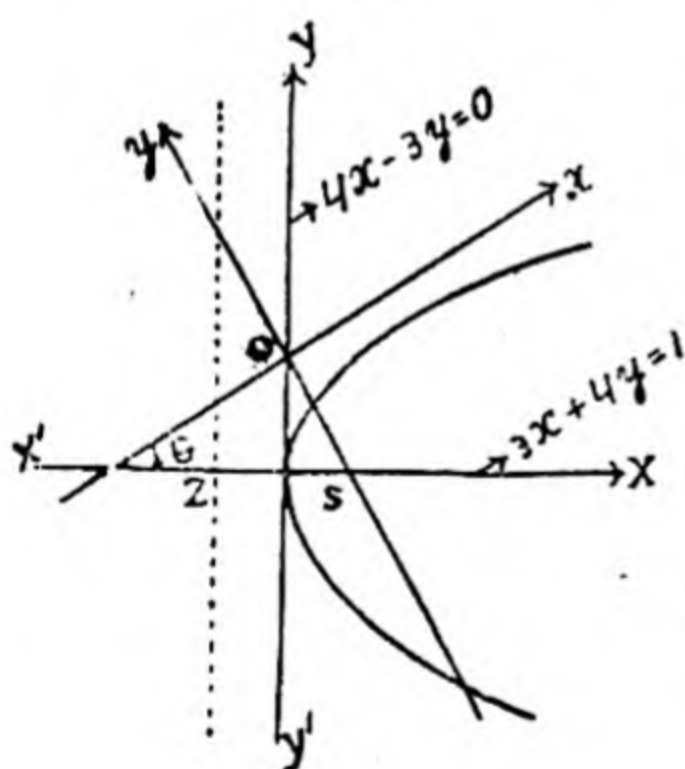


Fig. 18

It follows, therefore, that the parabola lies on the right of $Y'AX$, the tangent at the vertex as shown in fig. 18.

Hence, the expression for the perpendicular from any point on the parabola to the line $4x - 3y = 0$ should be preceded by the sign $+$.

With proper sign, the equation

(2) can, therefore, be written as

$$\left(\frac{3x + 4y + 1}{5}\right)^2 = \frac{2}{5} \left(\frac{4x - 3y}{5}\right) \quad \dots\dots(5)$$

The axis (3) of the parabola makes an angle θ with the x -axis such that $\tan \theta = -\frac{3}{4}$, i. e., $\sin \theta = -\frac{3}{5}$ and $\cos \theta = \frac{4}{5}$.

Now, the focus lies on the axis (3) at a distance from the vertex equal to one-fourth of the latus rectum; therefore, the coordinates of the focus are given by

$$\frac{x - (-3/25)}{4/5} = \frac{y - (-4/25)}{-3/5} = \frac{1}{4} \cdot \frac{2}{5}$$

Hence, the focus is the point $\left(-\frac{1}{25}, -\frac{11}{50}\right)$.

Again, the equation of the directrix is

$$4x - 3y = k, \quad \dots\dots(6)$$

where k is such that the length of the perpendicular from the vertex upon (6) is equal to one-fourth of the latus rectum;

$$\therefore \frac{0 - k}{\sqrt{4^2 + 3^2}} = \frac{1}{4} \cdot \frac{2}{5}, \text{ i. e., } k = \frac{1}{2}.$$

Hence, the equation of the directrix is

$$4x - 3y = \frac{1}{2}.$$

Note 1. If $3x + 4y + 1 = 0$ and $4x - 3y = 0$ be taken as the new axes of x and y respectively, the equation (5) is transformed into

$$Y^2 = \frac{2}{5}X. \quad \dots\dots(7)$$

Note 2. If in fig. 18, AX' and AY' are respectively taken as the new axes, equation (5) is changed into

$$Y^2 = -\frac{2}{5}X.$$

7.4. Central Conics. If the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad \dots\dots(1)$$

represents a central conic (an ellipse or a hyperbola), the coordinates of its centre are $(G/C, F/C)$, i. e., $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$ [see § 4.7].

The equation (1), when referred to parallel axes through the centre, becomes [§ 4.71]

$$ax^2 + 2hxy + by^2 + S_1 = 0,$$

or

$$Ax^2 + 2Hxy + By^2 = 1, \quad \dots\dots(2)$$

where $S_1 = gx_1 + fy_1 + c$ or $\frac{\Delta}{ab - h^2}$, $A = -a/S_1$, $H = -h/S_1$ and $B = -b/S_1$, (x_1, y_1) being the coordinates of the centre.

Hence, to investigate the central conics as regards lengths and position of the principal axes, eccentricity, foci etc., it is enough to consider an equation of the form (2) with centre at the origin.

7.41. To find the lengths and position of the principal axes of the central conic $ax^2 + 2hxy + by^2 = 1$ (1)

Method I. Consider the intersection of the conic (1) with a concentric circle of radius r whose equation is

$$x^2 + y^2 = r^2. \quad \dots\dots(2)$$

Subtracting $\frac{1}{r^2}$ times (2) from (1), we get the equation

$$\left(a - \frac{1}{r^2}\right)x^2 + 2hxy + \left(b - \frac{1}{r^2}\right)y^2 = 0, \quad \dots\dots(3)$$

which represents a pair of straight lines through the origin, i. e., the centre of the conic and the intersections of (1) and (2).

Now, the circle and the conic touch each other, if (2) passes through the extremities of either principal axis of (1) in which case r is the length of a semi-axis of the conic (1). Hence, if the lines (3) are coincident, they must lie along a principal axis.

The condition that (3) represents coincident lines

$$\left. \begin{aligned} \text{is} \quad & \left(a - \frac{1}{r^2}\right) \left(b - \frac{1}{r^2}\right) = h^2, \\ \text{i. e.,} \quad & (ab - h^2)r^4 - (a + b)r^2 + 1 = 0, \end{aligned} \right\} \quad \dots\dots(4)$$

which is a quadratic in r^2 and has two roots, say r_1^2 and r_2^2 .

If both r_1^2 and r_2^2 are positive and $r_1^2 > r_2^2$, then the conic is an ellipse with $2r_1$ and $2r_2$ as the lengths of its major and minor axes respectively.

If one of the values of r^2 , say r_1^2 , is positive, and r^2 is negative then the conic is a hyperbola with $2r_1$ and $2\sqrt{-r_2^2}$ as the lengths of its transverse and conjugate axes respectively.

Again, for values of r^2 given by (4), left hand member of (3) is a perfect square and equation (3) can be written as

$$\left[\left(a - \frac{1}{r^2}\right)x + hy\right]^2 = 0.$$

Hence, the equations of the principal axes corresponding to the roots r_1^2, r_2^2 of (4) are

$$\left(a - \frac{1}{r_1^2}\right)x + hy = 0, \quad \dots\dots(5)$$

and

$$\left(a - \frac{1}{r_2^2}\right)x + hy = 0. \quad \dots\dots(6)$$

If the conic is an ellipse and $r_1^2 > r_2^2$, then (5) is the equation of the major axis and (6) the equation of the minor axis.

If the conic is a hyperbola and $r_1^2 > 0, r_2^2 < 0$, then (5) is the equation of the transverse axis and (6) the equation of the conjugate axis.

Method II. The xy -term in (1) can be removed by rotating the axes through an angle θ such that $\tan 2\theta = \frac{2h}{a-b}$ and equation (1), when referred to its principal axes, becomes

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1, \quad \dots\dots(7)$$

Where r_2^2 is negative if the conic be a hyperbola.

By the theory of invariants (§ 3.13), we have

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = a + b, \text{ and } \frac{1}{r_1^2 r_2^2} = ab - h^2.$$

Hence, $\frac{1}{r_1^2}$ and $\frac{1}{r_2^2}$ are the roots of the quadratic in λ

$$\lambda^2 - \lambda(a+b) + ab - h^2 = 0,$$

i. e., r_1^2 and r_2^2 are the roots of the quadratic in r^2

$$(ab - h^2)r^4 - (a+b)r^2 + 1 = 0.$$

Again, the equation of the asymptotes (real or imaginary) of (7) is

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 0. \quad \dots\dots(8)$$

It is evident that the lines (8) make equal angles with either principal axis of (7). Hence, the principal axes of (1) are the pair of straight lines through the centre which bisect the angles between the asymptotes and their equation is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h},$$

i. e.,

$$h(x^2 - y^2) - (a - b)xy = 0. \quad \dots\dots(9)$$

These lines are always real, while the lines (8) may be real or imaginary.

Note. As the centre of the conic $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, is the point $(G/C, F/C)$ [§ 4.7], the equation of the axes of this conic is

$$h \left[(x - G/C)^2 - (y - F/C)^2 \right] - (a - b) (x - G/C) (y - F/C) = 0. \quad \dots\dots(10)$$

7.42. Eccentricity and Foci of the Central Conic

$$ax^2 + 2hxy + by^2 = 1. \quad \dots\dots(1)$$

The equation (1) when referred to its principal axes becomes

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1, \quad \dots\dots(2)$$

where r_1^2, r_2^2 are the roots of the equation (4) of § 7.41.

If the conic be an ellipse and $r_1 > r_2$, then the eccentricity e is given by

$$\begin{aligned} r_2^2 &= r_1^2 (1 - e^2); \\ \therefore e &= \sqrt{1 - r_2^2/r_1^2}. \end{aligned} \quad \dots\dots(3)$$

Further, the foci lie on the major axis (§ 7.41)

$$\left(a - \frac{1}{r_1^2} \right) x + hy = 0. \quad \dots\dots(4)$$

at distances $\pm er$, from the centre:

\therefore the coordinates of the foci of (1) are

$$(\pm er_1 \cos \theta, \pm er_1 \sin \theta), \quad \dots\dots(5)$$

where θ is the angle which (4) makes with the x -axis,

$$\text{i. e.,} \quad \tan \theta = - \left(a - \frac{1}{r_1^2} \right) / h.$$

If the conic be a hyperbola, the results (3) and (5) still hold, provided r_1 is the semi-transverse axis.

Note 1. The eccentricity e can be expressed in terms of a, b, h as follows :

We have

$$r_2^2 = r_1^2 (1 - e^2),$$

or

$$\frac{1}{1 - e^2} = \frac{r_1^2}{r_2^2},$$

Hence, the equations of the principal axes corresponding to the roots r_1^2, r_2^2 of (4) are

$$\left(a - \frac{1}{r_1^2} \right) x + hy = 0, \quad \dots\dots(5)$$

and

$$\left(a - \frac{1}{r_2^2} \right) x + hy = 0. \quad \dots\dots(6)$$

If the conic is an ellipse and $r_1^2 > r_2^2$, then (5) is the equation of the major axis and (6) the equation of the minor axis.

If the conic is a hyperbola and $r_1^2 > 0, r_2^2 < 0$, then (5) is the equation of the transverse axis and (6) the equation of the conjugate axis.

Method II. The xy -term in (1) can be removed by rotating the axes through an angle θ such that $\tan 2\theta = \frac{2h}{a-b}$ and equation (1), when referred to its principal axes, becomes

$$\frac{x^2}{r_1^2} + \frac{y^2}{r_2^2} = 1, \quad \dots\dots(7)$$

Where r_2^2 is negative if the conic be a hyperbola.

By the theory of invariants (§ 3.13), we have

$$\frac{1}{r_1^2} + \frac{1}{r_2^2} = a + b, \text{ and } \frac{1}{r_1^2 r_2^2} = ab - h^2.$$

Hence, $\frac{1}{r_1^2}$ and $\frac{1}{r_2^2}$ are the roots of the quadratic in λ

$$\lambda^2 - \lambda(a+b) + ab - h^2 = 0,$$

i. e., r_1^2 and r_2^2 are the roots of the quadratic in r^2

$$(ab - h^2)r^4 - (a+b)r^2 + 1 = 0.$$

Again, the equation of the asymptotes (real or imaginary)

of (7) is

$$\frac{x^2}{r_1^2} - \frac{y^2}{r_2^2} = 0. \quad \dots\dots(8)$$

It is evident that the lines (8) make equal angles with either principal axis of (7). Hence, the principal axes of (1) are the pair of straight lines through the centre which bisect the angles between the asymptotes and their equation is

$$\frac{x^2 - y^2}{a - b} = \frac{xy}{h},$$

i. e.,

$$h(x^2 - y^2) - (a - b)xy = 0. \quad \dots\dots(9)$$

$$\begin{aligned} \text{Since } \Delta = & \begin{vmatrix} 36 & 12 & -36 \\ 12 & 29 & 63 \\ -36 & 63 & 81 \end{vmatrix} = 108 \begin{vmatrix} 1 & 1 & -3 \\ 4 & 29 & 63 \\ -4 & 21 & 27 \end{vmatrix} \\ & = 108 \begin{vmatrix} 1 & 1 & -3 \\ 0 & 25 & 75 \\ 0 & 25 & 15 \end{vmatrix} \neq 0, \end{aligned}$$

and $ab - h^2 = 36 \times 39 - 12^2 > 0$, the curve is an *ellipse*.

Solving the equations

$$36x + 12y - 36 = 0 \quad \text{and} \quad 12x + 29y + 63 = 0,$$

we see that the centre is the point $C(2, -3)$.

The equation of the ellipse, when referred to parallel axes through $(2, -3)$, becomes

$$\begin{aligned} & 36x^2 + 24xy + 29y^2 - 36 \times 2 + 63 \times (-3) + 81 = 0, \\ \text{i.e.,} & \quad 36x^2 + 24xy + 29y^2 = 180, \\ \text{i.e.,} & \quad \frac{36}{180}x^2 + \frac{24}{180}xy + \frac{29}{180}y^2 = 1; \quad \dots\dots(1) \end{aligned}$$

\therefore the lengths of the semi-axes are given by

$$\begin{aligned} & \left(\frac{36}{180} \cdot \frac{29}{180} - \frac{12^2}{180^2} \right) r^2 - \left(\frac{36}{180} - \frac{29}{180} \right) r^2 + 1 = 0, \\ \text{i.e.,} & \quad \frac{1}{36} r^4 - \frac{13}{36} r^2 + 1 = 0, \quad \text{i.e. } r^4 - 13r^2 + 36 = 0, \\ \text{whence,} & \quad r_1^2 = 9, \text{ and } r_2^2 = 4. \quad \dots\dots(2) \end{aligned}$$

Hence the length of the semi-major axis is 3 and that of the semi-minor axis is 2.

The eccentricity e of the ellipse is given by

$$4 = 9(1 - e^2);$$

$$\therefore e = \sqrt{5/3}.$$

The equation of the major and minor axes referred to the centre $(2, -3)$ as origin are respectively

$$\left. \begin{aligned} & \left(\frac{36}{180} - \frac{1}{9} \right) x + \frac{12}{180} y = 0, \quad \text{i.e., } 4x + 3y = 0, \\ \text{and } & \left(\frac{36}{180} - \frac{1}{4} \right) x + \frac{12}{180} y = 0, \quad \text{i.e., } 3x - 4y = 0. \end{aligned} \right\} \dots\dots(3)$$

Referred to the old origin, these equations become

$$\left. \begin{array}{l} 4(x-2) + 3(y+3) = 0, \text{ i.e., } 4x + 3y + 1 = 0, \\ \text{and } 3(x-2) - 4(y+3) = 0, \text{ i.e., } 3x - 4y - 18 = 0. \end{array} \right\} \dots\dots (4)$$

The x -axis meets the ellipse, where

$$4x^2 - 8x + 9 = 0;$$

which shows that the points of intersection are imaginary.

Points of intersection with the y -axis are given by

$$29y^2 + 126y + 81 = 0,$$

whence $y = \frac{-63 \pm 10\sqrt{15}}{29} = 0.8$, or -3.5 approximately.

The intersections of the major axis with the ellipse are given

by
$$\frac{x-2}{3/5} = \frac{y+3}{-4/5} = \pm 3,$$

since the major axis $X'CX$ makes an angle θ (see fig. 19) with the x -axis such that $\tan \theta = -\frac{4}{3}$, i.e., $\cos \theta = \frac{3}{5}$ and $\sin \theta = -\frac{4}{5}$;

\therefore the points of intersection are

$$A \left(\frac{19}{5}, -\frac{27}{5} \right) \text{ and}$$

$$A' \left(\frac{1}{5}, -\frac{3}{5} \right)$$

Similarly, the intersections of the minor axis with the ellipse are given by

$$\frac{x-2}{4/5} = \frac{y+3}{3/5} = \pm 2;$$

\therefore the points of intersections are

$$B \left(\frac{18}{5}, -\frac{9}{5} \right) \text{ and}$$

$$B' \left(\frac{2}{5}, -\frac{21}{5} \right).$$

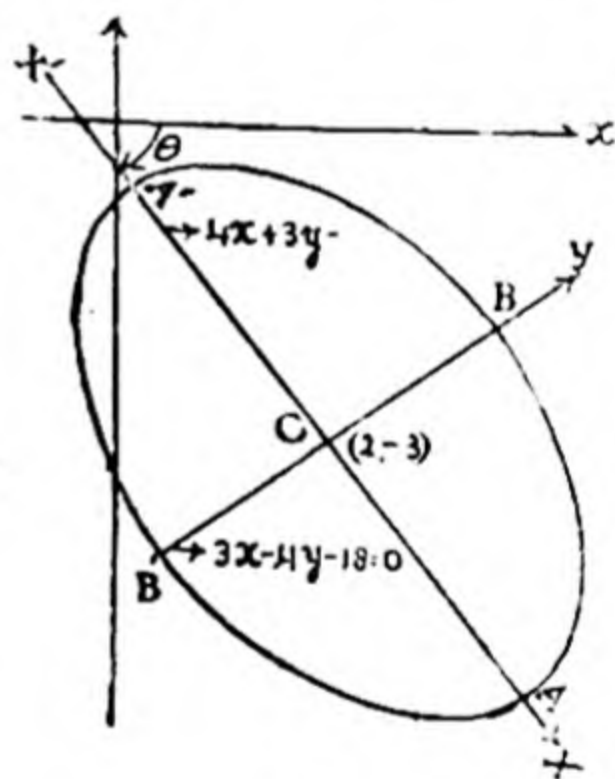


Fig. 19

Using the above information, the conic can be traced and the shape of the ellipse is as shown in fig. 19.

Note 1. The given equation when referred to CA as X -axis and CB as Y -axis becomes $\frac{X^2}{9} + \frac{Y^2}{4} = 1$.

Note 2. The foci are given by

$$\frac{x-2}{3/5} = \frac{y+3}{-4/5} = \pm 3 \times \frac{\sqrt{5}}{3} = \pm \sqrt{5};$$

\therefore the foci are the points $\left(2 \pm \frac{3}{\sqrt{5}}, -3 \pm \frac{4}{\sqrt{5}} \right)$.

The directrices meet the major axis in points given by

$$\frac{x-2}{3/5} = \frac{y+3}{-4/5} = \pm \frac{3}{\sqrt{5}/3} = \pm \frac{9}{\sqrt{5}},$$

i.e., the points of intersection are

$$\left(2 \pm \frac{27}{5\sqrt{5}}, -3 \mp \frac{36}{5\sqrt{5}} \right).$$

Hence, the equation of the latera recta and the directrices

are respectively $\frac{x-(2 \mp 3/\sqrt{5})}{4} = \frac{y-(-3 \pm 4/\sqrt{5})}{3},$

i.e., $3x-4y=18 \mp \frac{25}{\sqrt{5}}=18 \mp 5\sqrt{5},$

and $\frac{x-(2 \pm \frac{27}{5\sqrt{5}})}{4} = \frac{y-(-3 \mp \frac{36}{5\sqrt{5}})}{3},$

i.e., $3x-4y=18 \mp \frac{225}{5\sqrt{5}}=18 \mp 9\sqrt{5}.$

Ex. 2. Find the positions and lengths of the axes of the conic

$$x^2 - 4xy - 2y^2 + 10x + 4y = 0,$$

explaining the steps taken. Trace the conic. (Delhi '58; Vikram '60)

Here $\Delta = \begin{vmatrix} 1 & -2 & 5 \\ -2 & -2 & 2 \\ 5 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & -2 & 5 \\ 0 & -6 & 12 \\ 0 & 12 & -25 \end{vmatrix} \neq 0,$

and $ab-h^2 = -1 \times 2 - 2^2 < 0,$

Hence, the given conic is a *hyperbola*.

Solving the equations $x-2y+5=0$ and $-2x-2y+2=0$, we see that the centre is the point $C(-1, 2)$.

The equation of the conic when referred to parallel axes through $(-1, 2)$ becomes

$$x^2 - 4xy + 2y^2 + 5 \times (-1) + 2 \times 2 = 0,$$

i.e., $x^2 - 4xy - 2y^2 = 1; \quad \dots\dots(1)$

∴ the lengths of the semi-axes are given by

$$(-1 \cdot 2 - 2^2) r^4 - (1 - 2) r^2 + 1 = 0,$$

i.e.,

$$6r^4 - r^2 - 1 = 0,$$

whence

$$r_1^2 = \frac{1}{2}, r_2^2 = -\frac{1}{3}.$$

Hence, the length of semi-transverse axis is $\frac{1}{\sqrt{2}}$ and that of the semi-conjugate axis is $\frac{1}{\sqrt{3}}$.

The equations of the transverse and conjugate axes referred to the centre $(-1, 2)$ as origin are respectively

$$\text{and} \quad \left. \begin{aligned} (1-2)x - 2y &= 0, & \text{i.e., } x + 2y &= 0, \\ (1+3)x - 2y &= 0, & \text{i.e., } 2x - y &= 0. \end{aligned} \right\} \dots\dots(3)$$

Referred to the old axes, these equations become

$$\text{and} \quad \left. \begin{aligned} (x+1) + 2(y-2) &= 0, & \text{i.e., } x + 2y - 3 &= 0, \\ 2(x+1) - (y-2) &= 0, & \text{i.e., } 2x - y + 4 &= 0. \end{aligned} \right\} \dots\dots(4)$$

The x -axis meets the hyperbola,

where

$$x^2 + 10x = 0,$$

whence

$$x = 0, \text{ or } -10,$$

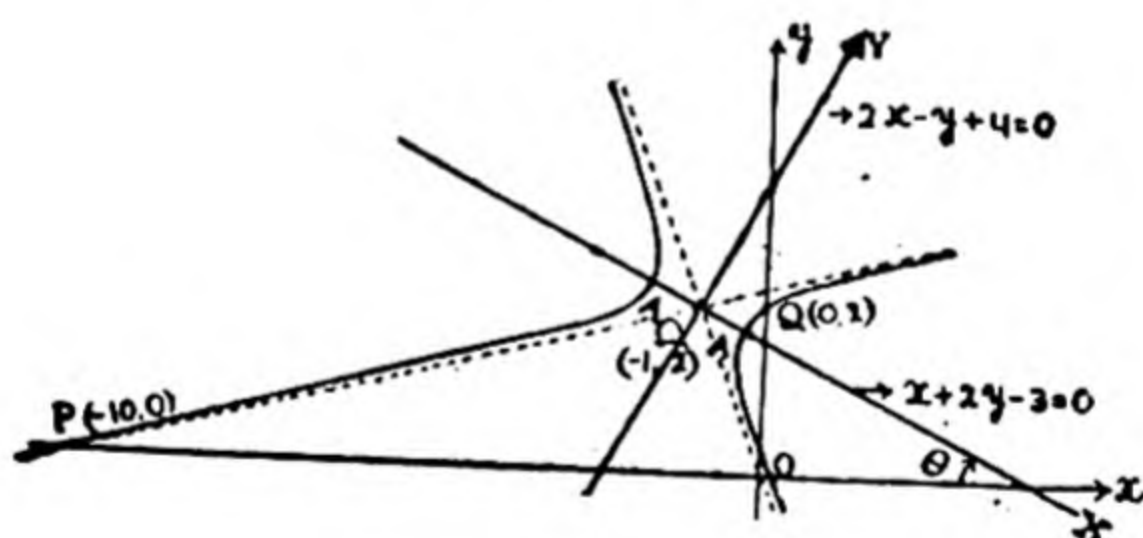


Fig. 20

i.e., the points of intersection with the x -axis are $O(0, 0)$ and $P(-10, 0)$.

The points of intersection with the y -axis are given by

$$-2y^2 + 4 = 0, \quad \text{whence } y = 0, \text{ or } 2,$$

i.e., the points of intersection are $O(0, 0)$ and $Q(0, 2)$.

The intersections of the transverse axis with the hyperbola

$$\text{are given by} \quad \frac{x+1}{2/\sqrt{5}} = \frac{y-2}{-1/\sqrt{5}} = \pm \frac{1}{\sqrt{2}},$$

i.e.,
$$\frac{x+1}{2} = \frac{y-2}{-1} = \pm 0.3 \text{ nearly,}$$

since the transverse axis makes an angle θ with the x -axis such that $\tan \theta = -\frac{1}{2}$, i.e., $\cos \theta = \frac{2}{\sqrt{5}}$ and $\sin \theta = -\frac{1}{\sqrt{5}}$;

\therefore the coordinates of the points of intersection A, A' are approximately $(-0.4, 1.7)$ and $(-1.6, 2.3)$.

The asymptotes of (1) are $x^2 - 4xy - 2y^2 = 0$,
i.e., $y = -0.4x$ and $y = 4.4x$, approximately.

Referred to the old axes, the equations of the asymptotes are $y - 2 = -0.4(x + 1)$ and $y - 2 = 4.4(x + 1)$, approximately.

Hence, the shape of the hyperbola is as shown in fig. 20.

Note 1. The equation of the hyperbola referred to its principal axes is $\frac{x^2}{1/2} - \frac{y^2}{1/3} = 1$, i.e., $2x^2 - 3y^2 = 1$.

Note 2. The eccentricity of the conic is given by

$$\frac{1}{2} = \frac{1}{2}(e^2 - 1);$$

$$\therefore e = \sqrt{\frac{5}{8}}.$$

Note 3. The negative value of r_2^2 also indicates that the conic is a hyperbola.

7.5. Miscellaneous Examples.

Ex. 1. Show that the length of the latus rectum of the parabola.

$$(a^2 + b^2)(x^2 + y^2) = (bx + ay - ab)^2$$

is
$$\frac{2ab}{\sqrt{a^2 + b^2}}.$$

(Bombay '47; Rajputana '51)

The given equation can be written as

$$(ax - by)^2 + (bx + ay)^2 = (bx + ay)^2 - 2ab(bx + ay) + a^2b^2,$$

or
$$\left(\frac{ax - by}{\sqrt{a^2 + b^2}} \right)^2 = \frac{-2ab}{\sqrt{a^2 + b^2}} (bx + ay - \frac{1}{2}ab),$$

which shows that it represents a parabola of latus rectum

$$\frac{2ab}{\sqrt{a^2 + b^2}}.$$

Ex. 2. Prove that the lengths of the semi-axes of the conic

$$ax^2 + 2hxy + ay^2 = d$$

are $\sqrt{\frac{d}{a+h}}$ and $\sqrt{\frac{d}{a-h}}$ respectively, and that their equation is $x^2 - y^2 = 0$. (Allahabad 1959)

The squares of the lengths of the semi-axes of the conic are the roots of the quadratic

$$\left(\frac{a}{d} - \frac{a}{d} - \frac{h^2}{d^2} \right) r^4 - \frac{2a}{d} r^2 + 1 = 0,$$

i.e., $(a^2 - h^2) r^4 - 2ad r^2 + d^2 = 0,$

whence $r^2 = \frac{ad \pm \sqrt{a^2 d^2 - (a^2 - h^2) d^2}}{a^2 - h^2}$

$$= \frac{(a \pm h) d}{a^2 - h^2}$$

$$= \frac{d}{a+h} \text{ or } \frac{d}{a-h}.$$

Hence, the lengths of the semi-axes are $\sqrt{\frac{d}{a+h}}$ and

$$\sqrt{\frac{d}{a-h}}.$$

The equations of the axes are

$$\left(\frac{a}{d} - \frac{a-h}{d} \right) x + \frac{h}{d} y = 0, \quad \text{i.e., } x+y=0,$$

and $\left(\frac{a}{d} - \frac{a+h}{d} \right) x + \frac{h}{d} y = 0, \quad \text{i.e., } x-y=0.$

The joint equation of the axes is

$$(x+y)(x-y)=0, \quad \text{i.e., } x^2 - y^2 = 0.$$

Or, The equation of the axes is [§ 7.41 (9)]

$$h(x^2 - y^2) - (a-a)xy = 0,$$

i.e., $x^2 - y^2 = 0.$

Ex. 3. Show that the coordinates of the foci of the conic

$$S \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are given by

$$\frac{(ax+hy+g)^2 - (hx+by+f)^2}{a-b} = \frac{(ax+hy+g)(hx+by+f)}{h} = S.$$

Let (α, β) be a focus of the conic, then the equation of the pair of tangents from it to the conic is (§ 4.4)

$$SS_1 = T^2,$$

$$\begin{aligned} \text{i.e., } (ax^2 + 2hxy + by^2 + 2gx + 2fy + c)(a\alpha^2 + 2h\alpha\beta + b\beta^2 + 2g\alpha + 2f\beta + c) \\ = [aax + h(\beta x + \alpha y) + b\beta y + g(x + \alpha) + f(y + \beta) + c]^2. \quad \dots (1) \end{aligned}$$

Since, (α, β) is a focus, equation (1) represents a point circle at (α, β) [§ 7.42, Note 2].

Now (1) represents a circle, if the coefficients of x^2 and y^2 are equal and the coefficient of xy is zero;

$$\begin{aligned} \therefore aS_1 - (a\alpha + h\beta + g)^2 &= bS_1 - (h\alpha + b\beta + f)^2, \\ \text{i.e., } \frac{(a\alpha + h\beta + g)^2 - (h\alpha + b\beta + f)^2}{a - b} &= S_1, \quad \dots (2) \end{aligned}$$

$$\begin{aligned} \text{and } hS_1 - (a\alpha + h\beta + g)(h\alpha + b\beta + f) &= 0, \\ \text{i.e., } \frac{1}{h}(a\alpha + h\beta + g)(h\alpha + b\beta + f) &= S_1 \quad \dots (3) \end{aligned}$$

Hence, it follows from (2) and (3) that the coordinates of the foci are given by

$$\frac{(ax + hy + g)^2 - (hx + by + f)^2}{a - b} = \frac{(ax + hy + g)(hx + by + f)}{h} = S.$$

Exercise 7

1. Trace the parabola

$$16x^2 - 24xy + 9y^2 + 77x - 64y + 95 = 0.$$

Also find the coordinates of its focus. (Agra 1950)

2. Find the axis, the vertex, the latus rectum and the focus of the parabola

$$16x^2 - 24xy + 9y^2 - 104x - 172y + 44 = 0.$$

Trace the conic. (Agra '55 S.; Delhi '57)

3. Trace the curve

$$17x^2 - 12xy + 8y^2 + 46x - 28y + 17 = 0.$$

Find the value of eccentricity and the equations of the axes. (Lucknow 1955)

Also find the equations of its directrices.

4. Trace the curve

$$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0,$$

and find its foci. (Agra '51, '55; Sagar '59)

5. Find the foci and the eccentricity of the conic

$$x^2 + 4xy + y^2 - 2x + 2y - 6 = 0.$$

Also trace it.

(Aligarh '58; Agra '59)

6. Trace the conic

$$32x^2 + 52xy - 7y^2 - 64x - 52y - 148 = 0;$$

Also find the coordinates of its foci.

(Allahabad '57; Rajputana '59)

7. Trace the conics :

(i) $9x^2 - 24xy + 16y^2 - 18x - 101y + 19 = 0.$

(Punjab '45; Kashmir '51)

(ii) $x^2 + 2xy + y^2 - 2x - 1 = 0.$

(Lucknow '52; S; Banaras '58)

(iii) $3(2x - 3y + 4)^2 + 2(3x + 2y - 5)^2 = 78$ (Vikram 1960)

(iv) $17x^2 + 12xy + 8y^2 - 46x - 28y + 33 = 0.$

(Aligarh '49; Agra '52)

(v) $97x^2 - 60xy + 72y^2 - 314x + 348y + 37 = 0.$

(Edinburgh 1946)

(vi) $x^2 - 3xy + y^2 + 10x - 10y + 21 = 0.$

(Agra 1958)

(vii) $108x^2 - 312xy + 17y^2 + 504x + 522y - 387 = 0.$

(I. A. A. S. 1940)

(viii) $4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0.$

(ix) $3x^2 + 8xy - 3y^2 - 40x - 20y + 50 = 0.$ (Kashmir 1951)

8. Show that the curve given by the equations

$$x = at^2 + bt + c, \quad y = a't^2 + b't + c'$$

is a parabola of latus rectum $(ab' - a'b)^2 / (a^2 + a'^2)^{3/2}$.

9. Show that the semi-axes of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are given by

$$(ab - h^2)^{3/2} r^4 + \Delta (a + b) (ab - h^2) r^2 + \Delta^2 = 0,$$

where Δ denotes the discriminant.

10. Prove that the squares of the semi axes of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

are $-2\Delta \div [(ab - h^2) (a + b \pm \sqrt{(a - b)^2 + 4h^2})]$.

where Δ is the discriminant.

11. Prove that the equation

$$\frac{(ax+hy+g)^2 - (hx+by+f)^2}{a-b} = \frac{(ax+hy+g)(hx+by+f)}{h}$$

represents the axes of the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0. \quad (\text{Delhi Hon's 1955})$$

12. Show that the product of the semi-axes of the conic whose equation is $(x-2y+1)^2 + (4x+2y-3)^2 - 10 = 0$, is 1.

13. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola, show that the equation referred to asymptotes as axes is

$$2(h^2 - ab)^{\frac{3}{2}} xy \pm \Delta = 0,$$

where Δ denotes the discriminant.

14. Find the equation of the hyperbola which has $3x-4y+7=0$ and $4x+3y+1=0$ for asymptotes and passes through the origin. Find its centre and trace the hyperbola.

(Agra 1957)

Answers

1. Axis $4x-3y+10=0$; tangent at the vertex $3x+4y+5=0$; vertex $\left(-\frac{11}{5}, \frac{2}{5}\right)$; latus rectum $4(3x+4y)+19=0$, length $\frac{1}{5}$; focus $\left(-\frac{217}{100}, \frac{11}{25}\right)$.

2. Axis $4x-3y+2=0$; tangent at the vertex $3x+4y-1=0$; vertex $\left(-\frac{1}{5}, \frac{2}{5}\right)$; latus rectum $3x+4y=11$, length 8; focus $(1, 2)$.

3. Ellipse; centre $(-1, 1)$; major axis $2x-y+3=0$, length 4; minor axis $x+2y-1=0$, length 2; eccentricity $\sqrt{3}/2$; directrices $x+2y-1 \pm 4\sqrt{\frac{5}{3}}=0$.

4. Ellipse; centre $(2, 3)$; major axis $2x-y+1=0$, length $2\sqrt{6}$; minor axis $x+2y-8=0$, length 4; eccentricity $1/\sqrt{3}$; foci $(2 \pm 2\sqrt{10}/5, 3 \pm \sqrt{10}/5)$.

5. Hyperbola; centre $(-1, 1)$; transverse axis $x-y+2=0$,

length $4\sqrt{3}$; conjugate axis $x+y=0$, length 4 ; asymptotes

$$x^2 + 4xy + y^2 - 2x + 2y - 2 = 0.$$

6. Hyperbola ; centre (1, 0) ; transverse axis $x-2y-1=0$, length 4 ; conjugate axis $2x+y-2=0$, length 6 ; asymptotes $8x-y-8=0$ and $4x+7y-4=0$;

foci $\left(1 \pm 2\sqrt{\frac{13}{5}}, \pm \sqrt{\frac{13}{5}} \right).$

7. (i) Parabola; axis $3x-4y+7=0$; tangent at the vertex $4x+3y+2=0$; vertex $\left(-\frac{29}{25}, \frac{22}{25}\right)$; length of latus rectum 3 ; the parabola is on the origin side of the tangent at the vertex.

(ii) Parabola ; axis $2x+2y-1=0$; tangent at the vertex $4x-4y+5=0$; vertex $\left(-\frac{3}{8}, \frac{7}{8}\right)$; latus rectum $1/\sqrt{2}$; the parabola is on the origin side of the tangent at the vertex.

(iii) Ellipse ; major axis $3x+2y-5=0$, length $2\sqrt{3}$; minor axis $2x-3y+4=0$, length $2\sqrt{2}$; centre $\left(-\frac{7}{13}, \frac{22}{13}\right).$

(iv) Ellipse ; centre (1, 1) ; major axis $2x+y-3=0$, length $4\sqrt{\frac{1}{5}}$; minor axis $x-2y+1=0$, length $2\sqrt{\frac{1}{5}}$.

(v) Ellipse ; centre (1, -2) ; major axis $3x-2y+7=0$, length 6 ; minor axis $2x+3y+4=0$, length 4.

(vi) Hyperbola ; centre (-2, 2) ; transverse axis $x-y+4=0$, length $2\sqrt{2}$; conjugate axis $x+y=0$, length $2\sqrt{\frac{2}{5}}$;

asymptotes $x^2-3xy+y^2+10x-10y+20=0.$

(vii) Hyperbola ; centre (2, 3) ; transverse axis $4x-3y+1=0$, length 6 ; conjugate axis $3x+4y-18=0$, length 4 ; asymptotes $6x-17y+39=0$, $18x-y-33=0.$

(viii) A pair of coincident straight lines.

(ix) Rectangular hyperbola ; centre (4, 2) ; transverse

axis $x-2y=0$, length $2\sqrt{10}$; conjugate axis $2x+y-10=0$;;
asymptotes $3x-y-10=0$ and $x+3y-10=0$.

15. $(3x-4y+7)(4x+3y+1)=7$; centre $(-1, 1)$; transverse axis $x+7y=6$, length $\frac{2}{5}\sqrt{14}$; conjugate axis $7x-y+8=0$, length $\frac{2}{5}\sqrt{14}$; the hyperbola is rectangular.

CHAPTER VIII

POLAR EQUATIONS

8.1. Polar Coordinates. The position of a point P in a plane can also be fixed by means of its distance from a fixed point O , called the **pole** (or the **origin**) and the angle made by OP (with proper sign) with a fixed straight line OX through the pole, called the **initial line** (or the **polar axis**).

The length OP is called the **radius vector** and is denoted by r . The angle XOP is called the **vectorial angle** and is denoted by θ . The numbers r and θ (with proper signs) are called the **polar coordinates** of P which is often referred to as the point (r, θ) . The radius vector is always written first.

Note. The straight line through the pole, perpendicular to the polar axis is called the **co-polar axis**.

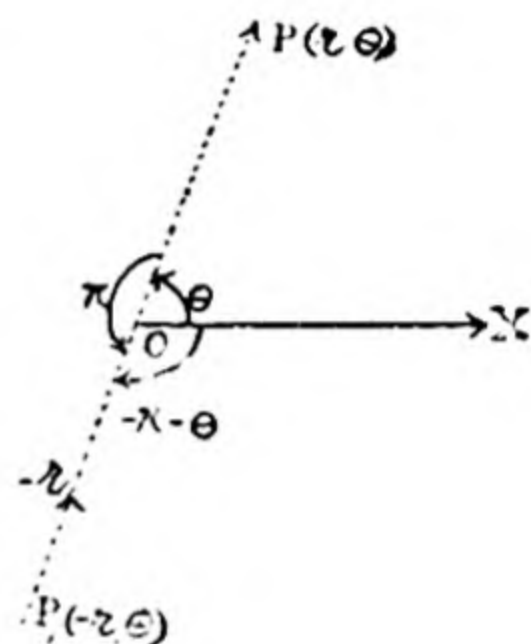


Fig. 21

8.11. The Convention of Signs. As in Trigonometry, the vectorial angle is regarded as positive if measured in the anti-clockwise direction and as negative if measured in the clockwise direction. The radius vector is regarded as positive if measured from the pole along the line bounding the vectorial angle and as negative if measured in the opposite direction (see fig. 21).

With this convention of signs, we see that if the polar coordinates of a point are given, the point is uniquely determined but the converse is not true. It is easy to see that the same point P can be represented in the following infinite number of ways :

$$(r, \theta \pm 2n\pi), n=0, 1, 2, 3, \dots; [-r, -(\pi - \theta) \pm 2n\pi], \\ n=0, 1, 2, 3, \dots; (-r, \pi + \theta \pm 2n\pi), n=0, 1, 2, 3, \dots$$

This shows the clear advantage of cartesian coordinates in this respect.

Note. In fig. 21, P_1 has the coordinates $(-r, \theta \pm 2n\pi)$, or $(r, \pi + \theta \pm 2n\pi)$, or $[r, -(\pi - \theta) \pm 2n\pi]$, where $n=0, 1, 2, 3, \dots$.

8.12. Relation between Polar and Rectangular Cartesian Coordinates. Let the pole O and the initial line OX in the polar system be the origin and the x -axis in the cartesian system and let the y -axis

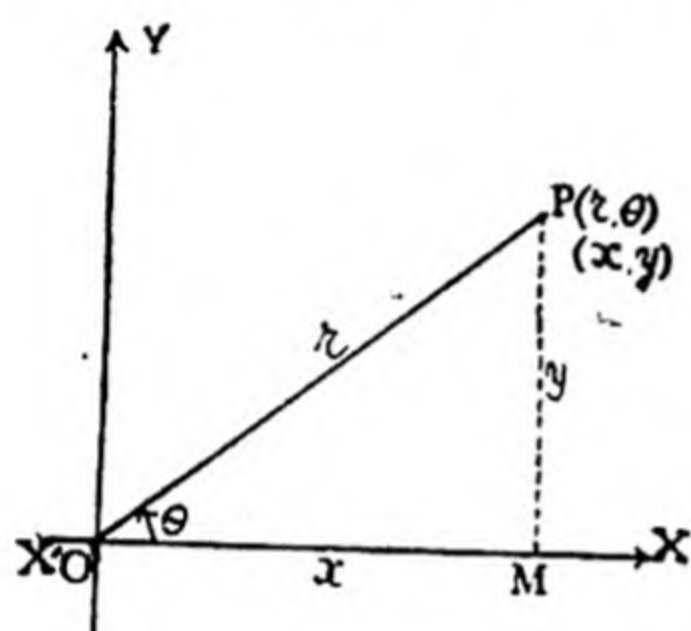


Fig. 22

makes a right angle with OX in the positive direction. Now, let (r, θ) and (x, y) be the polar and cartesian coordinates of P respectively.

If PM is perpendicular to OX , then in the triangle OMP (fig. 22),

$$x = r \cos \theta, \text{ and } y = r \sin \theta \dots\dots(1)$$

By squaring and adding the relations (1), and by division, we have, respectively

$$r^2 = x^2 + y^2, \text{ and } \theta = \tan^{-1} \frac{y}{x} \dots\dots(2)$$

The relations (1) and (2) can be used to transform an equation in rectangular coordinates into an equation in polar coordinates and *vice versa*.

8.13. Distance between the Points A (r_1, θ_1) and B (r_2, θ_2) .

From the $\triangle AOB$, we have

$$AB^2 = OA^2 + OB^2 - 2OA \cdot$$

$$OB \cos \angle AOB$$

$$= r_1^2 + r_2^2 - 2r_1r_2 \cos (\theta_2 - \theta_1),$$

$$\text{i.e., } AB = \pm \sqrt{[r_1^2 + r_2^2 - 2r_1r_2 \cos (\theta_2 - \theta_1)]} \dots\dots(1)$$

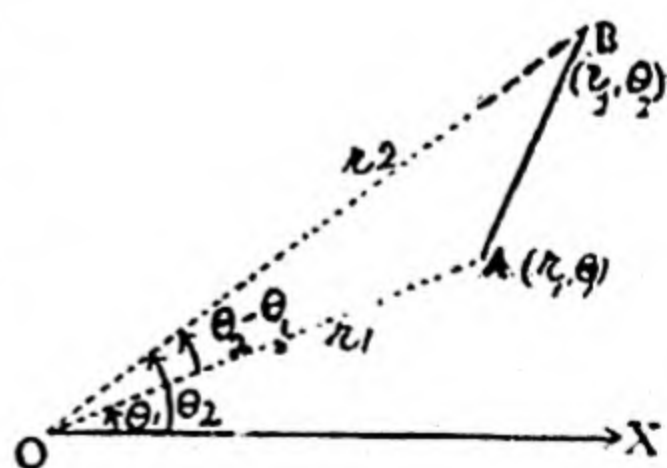


Fig. 23

Note. The result (1) above is independent of the order in which points are taken, since $\cos (\theta_2 - \theta_1) = \cos (\theta_1 - \theta_2)$.

8.14. Area of a Triangle in Polar Coordinates.

Let $A (r_1, \theta_1)$, $B (r_2, \theta_2)$ and $C (r_3, \theta_3)$ be the vertices of a triangle (fig. 24), then

$$\begin{aligned}\triangle ABC &= \triangle OAB + \triangle OBC - \triangle OAC \\ &= \frac{1}{2} OA \cdot OB \sin \angle AOB \\ &\quad + \frac{1}{2} OB \cdot OC \sin \angle BOC \\ &\quad - \frac{1}{2} OA \cdot OC \sin \angle AOC\end{aligned}$$

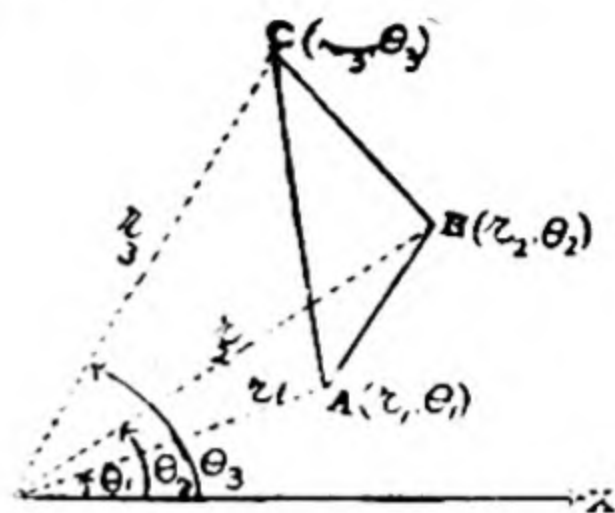


Fig. 24

$$\begin{aligned}&= \frac{1}{2} [r_1 r_2 \sin (\theta_2 - \theta_1) + r_2 r_3 \sin (\theta_3 - \theta_2) - r_1 r_3 \sin (\theta_3 - \theta_1)], \\ \text{i.e., } &= \frac{1}{2} [\mathbf{r_1 r_2 \sin (\theta_2 - \theta_1) + r_2 r_3 \sin (\theta_3 - \theta_2) + r_3 r_1 \sin (\theta_1 - \theta_3)}].\end{aligned}$$

Ex. 1. Transform the equation $x^2 - y^2 = 2ay$ to polar coordinates.

Putting $x = r \cos \theta$, $y = r \sin \theta$ in the given equation, we have

$$r^2 (\cos^2 \theta - \sin^2 \theta) = 2ar \sin \theta,$$

or

$$r \cos 2\theta = 2a \sin \theta,$$

which is the required equation.

Ex. 2. Transform the equation $r^{\frac{1}{2}} \cos \frac{\theta}{2} = a^{\frac{1}{2}}$ to rectangular coordinates.

Squaring both sides of the given equation, we have

$$r \cos^2 \frac{\theta}{2} = a,$$

or

$$r (1 + \cos \theta) = 2a,$$

or

$$r = 2a - r \cos \theta = 2a - x, \text{ putting } r \cos \theta = x.$$

Squaring again, we get

$$r^2 = (2a - x)^2,$$

i.e.,

$$x^2 + y^2 = (2a - x)^2, \text{ putting } r^2 = x^2 + y^2,$$

or

$$y^2 + 4ax = 4a^2, \text{ which is the required equation.}$$

8.2. Polar Equation of a Straight Line. The general equation of a straight line in cartesian coordinates is

$$ax + by + c = 0. \quad \dots\dots(1)$$

Writing $x=r \cos \theta$, $y=r \sin \theta$, the equation (1) transforms into

$$r(a \cos \theta + b \sin \theta) + c = 0, \quad \dots\dots(2)$$

which is the most general equation of a straight line in polar coordinates.

If $c \neq 0$, i. e., the line (1) does not pass through the origin, then (2) can be written as

$$-\frac{a}{c} \cos \theta + \frac{-b}{c} \sin \theta = \frac{1}{r},$$

i.e., $A \cos \theta + B \sin \theta = \frac{1}{r}. \quad \dots\dots(3)$

Similarly, other forms of the equation of a straight line in cartesian coordinates can be transformed to the polar form by putting $x=r \cos \theta$ and $y=r \sin \theta$.

We shall now derive two important polar forms independently.

I. Normal Form.

Let AB be the given straight line, OX the initial line, OL the perpendicular from the pole on AB , such that $OL=p$ and $\angle XOL=a$.

Let $P(r, \theta)$ be any point on AB , the $OP=r$ and $\angle XOP=\theta$ so that $\angle LOP=\theta-a$.

Now, from the right-angled

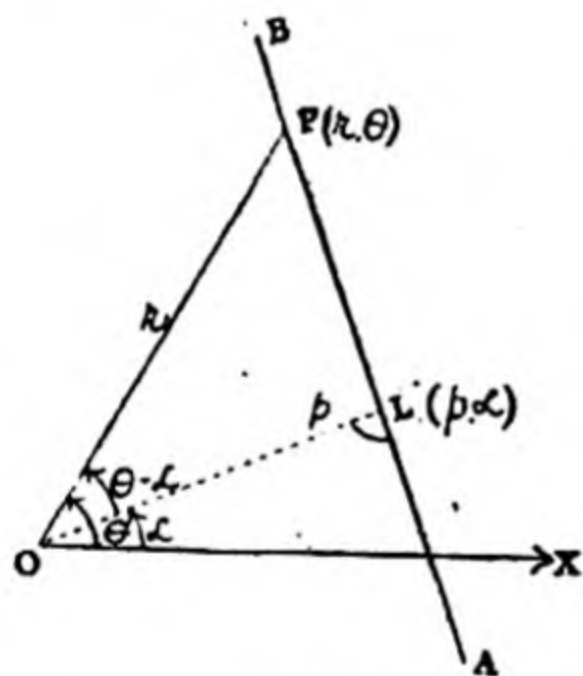


Fig. 25

triangle OLP , we have (fig. 25)

$$\frac{OL}{OP} = \cos \angle LOP, \text{ i.e., } \frac{p}{r} = \cos (\theta - a).$$

or $p = r \cos (\theta - a), \quad \dots\dots(4)$
which is the required equation,

II. Two-Point Form.

Let $A(r_1, \theta_1)$, $B(r_2, \theta_2)$ be two given points on the straight line and $P(r, \theta)$ any point on it, then $\triangle AOP + \triangle POB = \triangle AOB$ (fig. 26),

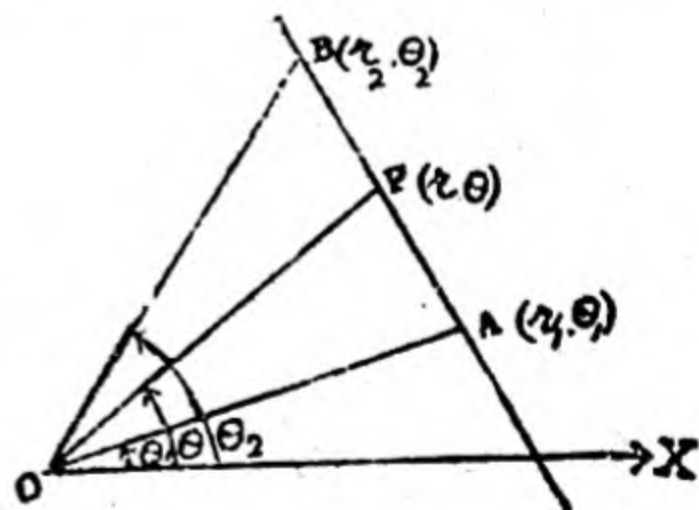


Fig. 26

$$\text{i.e., } \frac{1}{2} r r_1 \sin (\theta - \theta_1) + \frac{1}{2} r r_2 \sin (\theta_2 - \theta) = \frac{1}{2} r_1 r_2 \sin (\theta_2 - \theta_1),$$

$$\text{or } \frac{\sin (\theta - \theta_1)}{r_2} + \frac{\sin (\theta_2 - \theta)}{r_1} = \frac{\sin (\theta_2 - \theta_1)}{r},$$

$$\text{or } \frac{\sin (\theta_2 - \theta_1)}{r} + \frac{\sin (\theta_1 - \theta)}{r_2} + \frac{\sin (\theta - \theta_2)}{r_1} = 0, \dots (1)$$

which is the required equation.

Note. 1. When the polar equation of a straight line is known in the form (4), the coordinates of the foot of the perpendicular from the pole are immediately known to be (p, a) .

Note. 2. A straight line parallel to (4) is given by

$$k = r \cos (\theta - a),$$

where k is a parameter, and a st. line perpendicular to (4) by

$$p' = r \cos [\theta - (a \pm \pi/2)] \\ = \pm r \sin (\theta - a),$$

$$\text{or } l = r \sin (\theta - a), \text{ where } l \text{ is a parameter. } \dots (6)$$

Thus, (6) can be obtained from (4) by changing p to l and θ to $\theta - \pi/2$.

Ex. Find the polar coordinates of the foot of the perpendicular from the pole on the straight line joining the two points (r_1, θ_1) and (r_2, θ_2) .
(Punjab 1946)

The equation of the st. line joining the two points is

$$\frac{\sin (\theta_2 - \theta_1)}{r} + \frac{\sin (\theta_1 - \theta)}{r_2} + \frac{\sin (\theta - \theta_2)}{r_1} = 0,$$

$$\text{or } r_1 r_2 \sin (\theta_1 - \theta_2) = r [r_1 \sin (\theta_1 - \theta) + r_2 \sin (\theta - \theta_2)] \\ = r [(r_1 \sin \theta_1 - r_2 \sin \theta_2) \cos \theta \\ + (r_2 \cos \theta_2 - r_1 \cos \theta_1) \sin \theta] \\ = k r \cos (\theta - a), \dots (1)$$

$$\text{where } k = \sqrt{[(r_1 \sin \theta_1 - r_2 \sin \theta_2)^2 + (r_2 \cos \theta_2 - r_1 \cos \theta_1)^2]} \\ = \sqrt{[r_1^2 + r_2^2 - 2 r_1 r_2 \cos (\theta_1 - \theta_2)]}, \dots (2)$$

$$\text{and } a = \tan^{-1} \frac{r_2 \cos \theta_2 - r_1 \cos \theta_1}{r_1 \sin \theta_1 - r_2 \sin \theta_2}. \dots (3)$$

Hence, the coordinates of the foot of the perpendicular from the pole on the line (1) are $[\frac{1}{k} r_1 r_2 \sin (\theta_1 - \theta_2), a]$, where k and a are given by (2) and (3).

8.3. Polar Equation of a circle.

I. General Form.

Let O be the pole, OX the initial line, $C(c, \alpha)$ the centre and a the radius of the circle. Also, let $P(r, \theta)$ be any point on the circle (fig. 27), then from the triangle OCP , we have

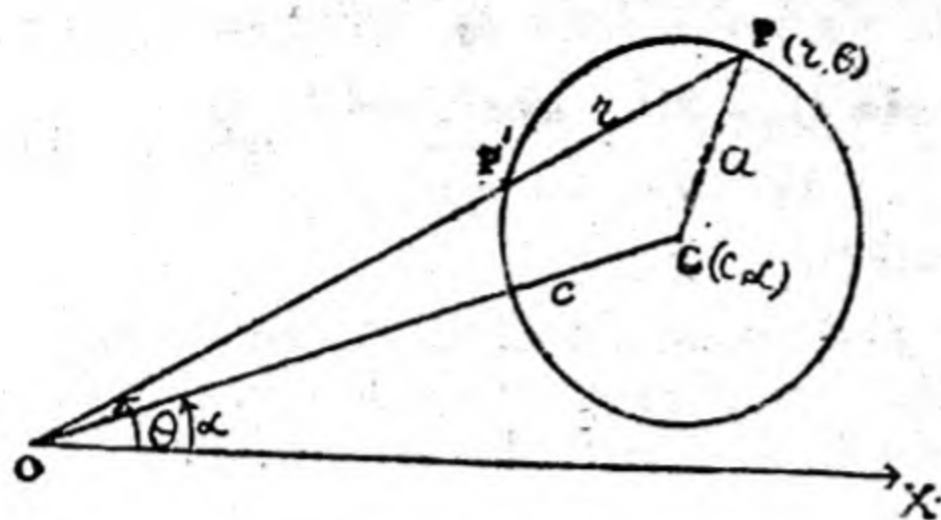


Fig. 27

$CP^2 = OC^2 + OP^2 - 2OC \cdot OP \cos \angle COP$,
i.e., $a^2 = c^2 + r^2 - 2cr \cos(\theta - \alpha)$, which is the equation of the given circle. This equation is usually written as

$$r^2 - 2rc \cos(\theta - \alpha) + (c^2 - a^2) = 0. \quad \dots\dots(1)$$

Note. If the radius vector OP meets the circle in another point P' , P' can also be taken as (r, θ) .

II. Particular Forms. (i) If the circle above passes through the pole, then $c = a$ and the equation (1) reduces to

$$r = 2a \cos(\theta - \alpha). \quad \dots\dots(2)$$

(ii) If in addition the centre C lies on the initial line, then $\alpha = 0$ and the equation reduces to

$$r = 2a \cos \theta. \quad \dots\dots(3)$$

This is the simplest form of the polar equation of a circle and can be derived independently quite easily.

Since C lies on the initial line, OA is a diameter, where A is the point in which the circle meets OX (fig. 28). If $P(r, \theta)$ is any point on the circle, then from the right-angled triangle OAP , we have

$$OP = OA \cos \theta.$$

i.e., $r = 2a \cos \theta$, [$\because OA = 2OC = 2a$].

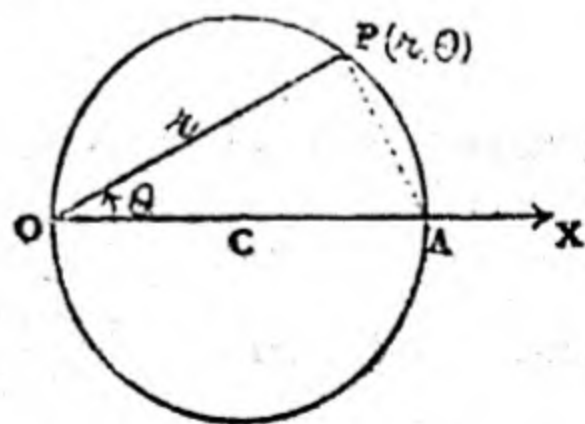


Fig. 28

(iii) If C lies on OX but the circle does not pass through O , then $\alpha = 0$ and the equation (1) reduces to

$$r^2 - 2cr \cos \theta + c^2 - a^2 = 0. \quad \dots\dots(4)$$

(iv) If C is at O , $c=0$, and the equation (1) reduces to

$$\mathbf{r}=\mathbf{a}, \quad \dots\dots(5)$$

which is obvious.

8.31. Polar Equation of a Circle on a Given Diameter.

Let $A(a, \alpha)$ and $B(b, \beta)$ be the extremities of a given diameter, then if $P(r, \theta)$ is any point on the circle having AB as a diameter, $\angle APB$ is a right angle; hence

$$AP^2 + BP^2 = AB^2,$$

$$\text{or} \quad r^2 + a^2 - 2ra \cos(\theta - \alpha) + r^2 + b^2 - 2rb \cos(\theta - \beta) = a^2 + b^2 + 2ab \cos(\alpha - \beta),$$

$$\text{or} \quad r^2 - r[a \cos(\theta - \alpha) + b \cos(\theta - \beta)] + ab \cos(\alpha - \beta) = 0,$$

which is the equation of the given circle.

8.32. Polar Equation of a chord joining Two Given Points.

Let A and B be the given points on the circle

$$r = 2a \cos \theta,$$

and let their vectorial angles be α and β so that their coordinates are $(2a \cos \alpha, a)$ and $(2a \cos \beta, \beta)$ respectively.

Now any straight line in polar coordinates is

$$p = r \cos(\theta - \theta_1). \quad \dots\dots(1)$$

This passes through A and B , if

$$p = 2a \cos \alpha \cos(\alpha - \theta_1) = a [\cos(2\alpha - \theta_1) + \cos \theta_1], \quad \dots\dots(2)$$

$$\text{and} \quad p = 2a \cos \beta \cos(\beta - \theta_1) = a [\cos(2\beta - \theta_1) + \cos \theta_1], \quad \dots\dots(3)$$

$$\text{i.e., if } \cos(2\alpha - \theta_1) + \cos \theta_1 = \cos(2\beta - \theta_1) + \cos \theta_1,$$

$$\text{i.e., if } 2\alpha - \theta_1 = -(2\beta - \theta_1) \quad [\because \alpha \neq \beta],$$

$$\text{i.e., if } \theta_1 = \alpha + \beta, \text{ whence from (2) or (3), } p = 2a \cos \alpha \cos \beta.$$

Hence, the equation of the chord AB is

$$2a \cos \alpha \cos \beta = r \cos(\theta - \alpha - \beta). \quad \dots\dots(4)$$

Cor. Making $\beta \rightarrow \alpha$ in (4), we get the equation of the tangent at A as

$$2a \cos^2 \alpha = r \cos(\theta - 2\alpha). \quad \dots\dots(5)$$

Ex. Find the polar equation of circles passing through the points whose polar coordinates are $(a, \pi/2)$, $(b, \pi/2)$ and touching the straight line $\theta = 0$.

Let the equation of a circle satisfying the given condition be

$$r^2 - 2r(A \cos \theta + B \sin \theta) + C = 0^* \quad \dots\dots(1)$$

This touches the st. line $\theta=0$, if $r^2 - 2rA + C = 0$ has equal roots, i.e., if $C = A^2$. Hence, the equation (1) becomes

$$r^2 - 2r(A \cos \theta + B \sin \theta) + A^2 = 0. \quad \dots\dots(2)$$

This passes through $(a, \pi/2)$ and $(b, \pi/2)$, if

$$a^2 - 2aB + A^2 = 0, \quad \dots\dots(3)$$

and

$$b^2 - 2bB + A^2 = 0. \quad \dots\dots(4)$$

From (3) and (4), by subtraction, we have

$$2(a-b)B = a^2 - b^2, \text{ or } 2B = a + b.$$

Substituting this value in (3) or (4), we get

$$a^2 - a(a+b) + A^2 = 0, \text{ or } A^2 = ab, \text{ i.e., } A = \pm\sqrt{ab}.$$

Hence, the two circles satisfying the given conditions are

$$r^2 \pm 2\sqrt{ab} r \cos \theta - (a+b) r \sin \theta + ab = 0.$$

***Note.** The equation (1) of § 8.3 can be written in this form.

Transforming to cartesian coordinates, this equation becomes

$$x^2 + y^2 - 2Ax - 2By + C = 0,$$

with which we are already familiar.

8.4. Polar Equations of Conics.

Putting $x = r \cos \theta$, $y = r \sin \theta$ in the equation

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,$$

we see that the general equation of a conic in polar coordinates is

$$r^2(a \cos^2 \theta + 2h \cos \theta \sin \theta + b \sin^2 \theta) + 2r(g \cos \theta + f \sin \theta) + c = 0. \quad \dots\dots(1)$$

Similarly, the polar equation of the ellipse corresponding to the cartesian equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$r^2 \left(\frac{\cos^2 \theta}{a^2} + \frac{\sin^2 \theta}{b^2} \right) = 1; \quad \dots\dots(2)$$

the polar equation of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is

$$r^2 \left(\frac{\cos^2 \theta}{a^2} - \frac{\sin^2 \theta}{b^2} \right) = 1; \quad \dots\dots(3)$$

and the polar equation of the parabola $y^2 = 4ax$ is

$$r^2 \sin \theta = 4a \cos \theta \quad \dots\dots(4)$$

The equations (1), (2), (3) and (4), however, are not much used. We need the use of polar coordinates generally in problems on focal distances. In such cases a focus is taken to be the pole and the equation of the conic assumes a simple form.

8.41. To find the Polar equation of a conic, when the pole is at a focus and the initial line is perpendicular to the directrix.

Let S be a focus and DZ the corresponding directrix of a conic of eccentricity e and let SZ be the perpendicular from S on the directrix DZ (fig. 29).

Now, take S as the pole and SZ as the initial line and let $P(r, \theta)$ be any point on the conic, then $SP=r$, and $\angle ZSP=\theta$.

Draw PM , PN perpendiculars respectively to DZ , SZ and let the semi-latus rectum $SL=l$, then

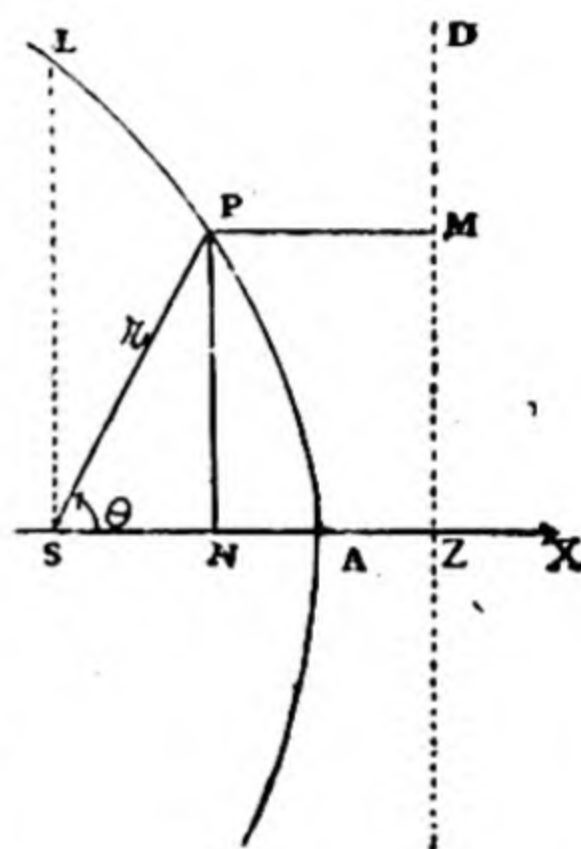


Fig. 29

$$SP = e \cdot PM \quad (\text{by the definition of a conic})$$

$$= e \cdot NZ = e(SZ - SN)$$

$$= e \cdot \left(\frac{l}{e} - r \cos \theta \right) \quad [\because SL = e \cdot SZ]$$

$$\therefore r = l - e r \cos \theta,$$

$$\text{i.e.,} \quad \frac{l}{r} = 1 + e \cos \theta, \quad \dots\dots(1)$$

which is the required equation.

Cor. 1. If, however, the axis SZ of the conic makes an angle α with the initial line, then the equation of the conic takes the form

$$\frac{l}{r} = 1 + e \cos (\theta - \alpha); \quad \dots\dots(2)$$

for, in this case, the angle $\angle ZSP$ is $\theta - \alpha$.

Cor. 2. If the initial line is taken in the direction ZS

instead of SZ , then the equation (1) becomes

$$\frac{l}{r} = 1 - e \cos \theta ; \quad \dots\dots(3)$$

for, in this case, the angle ZSP is $\pi - \theta$.

Note 1. There is no agreement among the various writers on the subject as regards the standard equation of a conic in polar coordinates. Some writers use the form (1) while others use the form (3).

We prefer the form (1) as it is the one most commonly used in Coordinate Geometry as well as Astronomy.

The reader should, however, try to become equally familiar with both the forms.

Note 2. When $\theta = a$, $r = \frac{l}{1 + e \cos a}$, from (1).

Hence, the polar coordinates of any point on the conic (1) are $\left(\frac{l}{1 + e \cos a}, a \right)$. This point is usually referred to as the point 'a'.

8.42. Directrices of $\frac{1}{r} = 1 + e \cos \theta$.

Case I. When the conic is an ellipse.

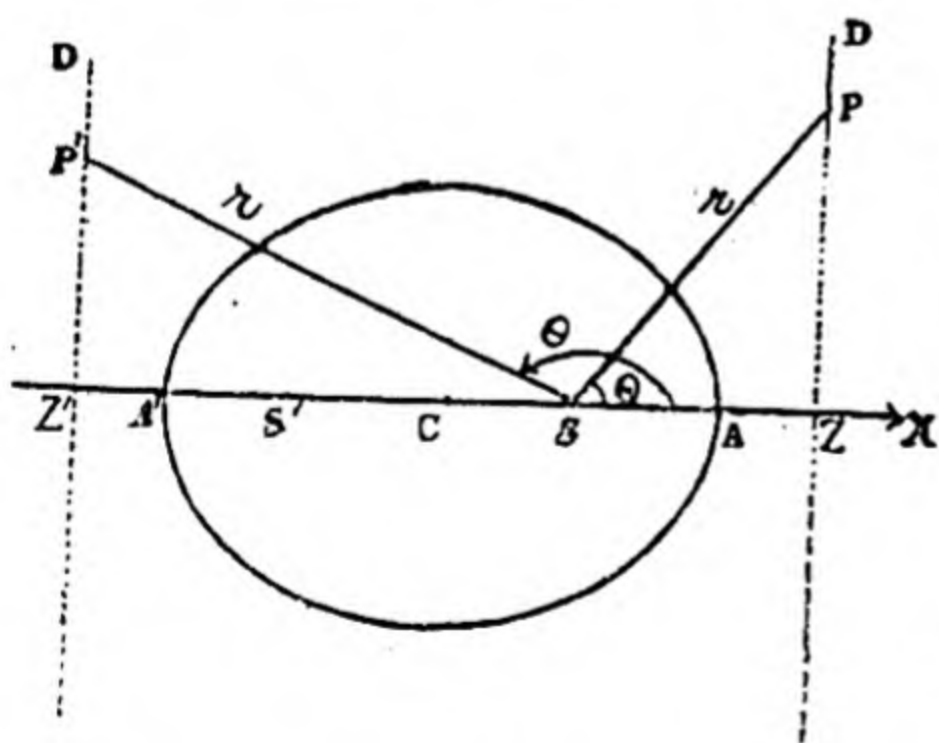


Fig. 30

If $P(r, \theta)$ is any point on the directrix DZ corresponding to the focus S (fig. 30) which has been taken as the pole, then,

$$r \cos \theta = SZ = \frac{l}{e} ;$$

\therefore the equation of the directrix DZ is

$$\frac{1}{r} = e \cos \theta \quad \dots\dots(1)$$

If $P(r, \theta)$ is any point on the directrix $D'Z'$ corresponding to the other focus S' , then

$r \cos (\pi - \theta) = SZ' = SC + CZ'$, where C is the centre

$= a e + \frac{a}{e}$, where a is the semi-major axis

$$= \frac{a}{e} (e^2 + 1). \quad \dots\dots(2)$$

Now, $l = \frac{b^2}{a} = a (1 - e^2)$;

$$\therefore a = \frac{l}{1 - e^2}$$

Hence, equation (2) becomes

$$-r \cos \theta = \frac{l}{e} \cdot \frac{1 + e^2}{1 - e^2},$$

$$\text{i.e.} \quad \frac{l}{r} + \frac{e(1 - e^2)}{1 + e^2} \cos \theta = 0, \quad \dots\dots(3)$$

which is the equation of the directrix $D'Z'$.

Case II. When the conic is a hyperbola.

Proceeding as above, we see that equations (1) and (3) hold in this case also.

Case III. When the conic is a parabola, i.e., $e = 1$.

In this case, the parabola $\frac{l}{r} = 1 + \cos \theta$ has only one directrix at a finite distance whose equation is easily seen to

$$\text{be} \quad \frac{l}{r} = \cos \theta. \quad \dots\dots(4)$$

The other directrix lies at infinity.

$$\mathbf{8.43. \quad Tracing the Conic} \quad \frac{l}{r} = 1 + e \cos \theta. \quad \dots\dots(1)$$

Case I. Let $e = 1$, then the conic is a parabola.

The equation (1), in this case, becomes

$$\frac{l}{r} = 1 + \cos \theta = 2 \cos^2 \frac{\theta}{2},$$

$$\text{i.e.,} \quad r = \frac{l}{2 \cos^2 \frac{\theta}{2}}. \quad \dots\dots(2)$$

The conic (2) meets the initial line $\theta = 0$ in the point A , such that $r = \frac{l}{2}$, i.e. the coordinates of the vertex A are

$$\left(\frac{l}{2}, 0 \right).$$

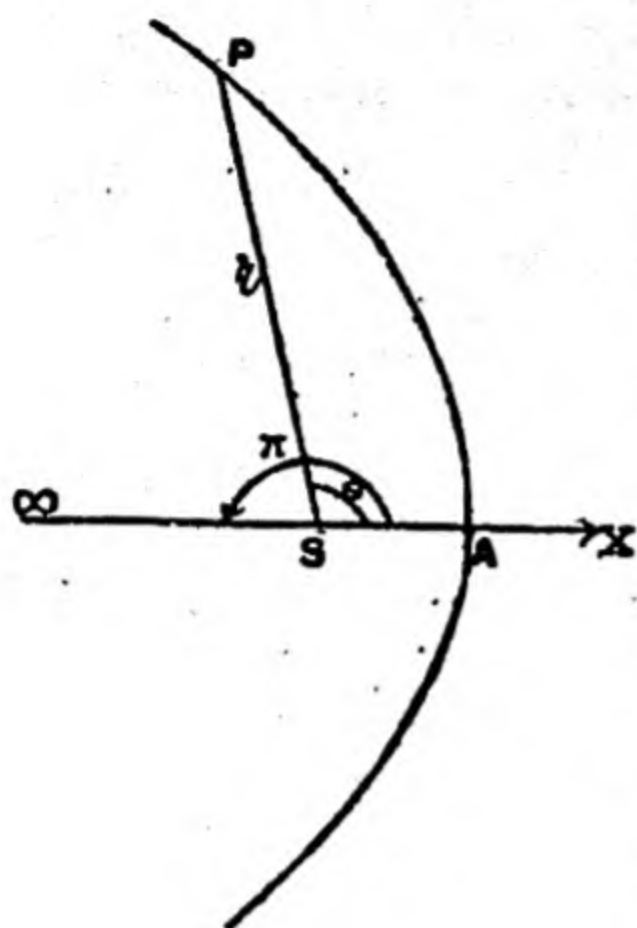


Fig. 31

Hence, as θ increases beyond π , the values of r are repeated. So the shape of the curve is as shown in fig. 31.

Case II. Let $e < 1$, then the conic is an ellipse.

The conic (1) meets the initial line $\theta = 0$, where

$$\frac{l}{r} = 1 + e, \text{ i.e., } r = \frac{l}{1+e};$$

\therefore the point A (fig. 30) is $\left(\frac{l}{1+e}, 0\right)$.

As θ increases from 0 to π , $\cos \theta$ decreases continuously and, therefore, r increases. When $\theta = \pi$, $r = \frac{l}{1-e} > 0$, since $e < 1$. The curve, therefore, meets the initial line in another point A' , such that $SA' = \frac{l}{1-e}$.

Also, the curve is symmetrical about the initial line, since $\cos(-\theta) = \cos \theta$. Hence, the curve is a closed one and its shape is as shown in fig. 30.

Case. III. Let $e < 1$, then the conic is a hyperbola.

The conic (1) meets the initial line $\theta = 0$, where

$$r = -\frac{l}{1+e},$$

The minimum value of r is obtained when $\theta = 0$, as in this case

$\cos^2 \frac{\theta}{2}$ is maximum.

As θ increases from 0 to π , $\cos^2 \frac{\theta}{2}$ decreases continuously and, therefore, r increases. As $\theta \rightarrow \pi$, $\cos^2 \frac{\theta}{2} \rightarrow 0$ and $r \rightarrow \infty$.

On changing θ to $-\theta$, we see that r remains unchanged :

\therefore the curve is symmetrical about the initial line.

so that A is the point

$$\left(-\frac{l}{1+e}, 0\right).$$

As θ increases from 0 to $\pi/2$, $\cos \theta$ decreases from 1 to 0 and, therefore, r increases from $\frac{l}{1+e}$ to l . As θ increases beyond $\pi/2$, $\cos \theta$ becomes negative and r becomes still

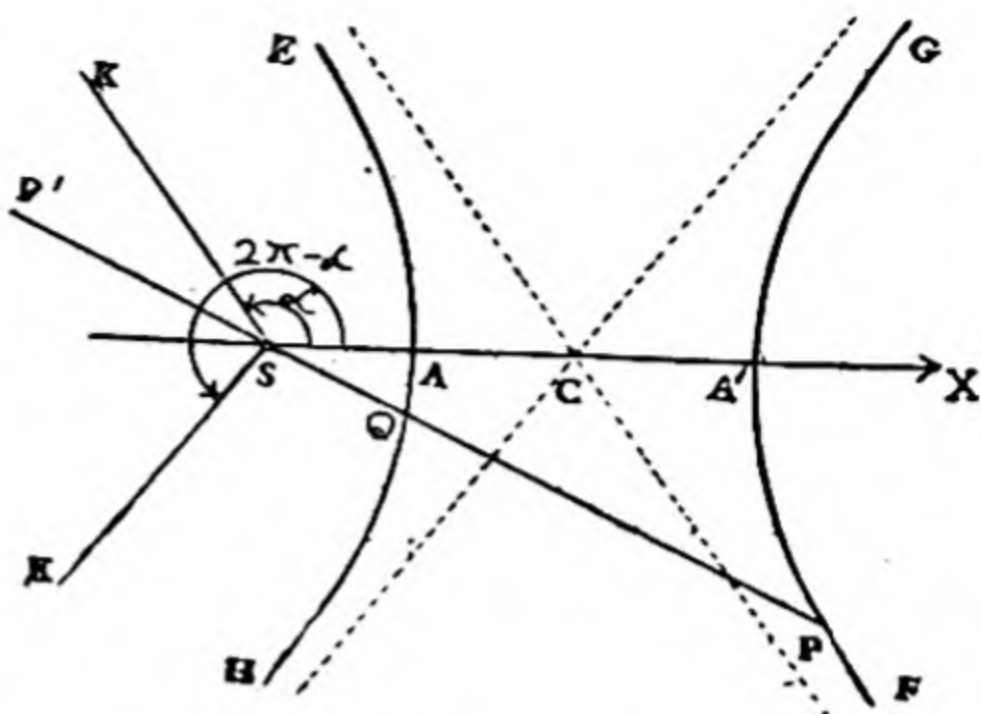


Fig. 32

greater and goes on increasing until $1 + e \cos \theta = 0$,

i.e., $\theta = \cos^{-1} \left(-\frac{1}{e} \right) = \alpha$, say. When $\theta \rightarrow \alpha$, $r \rightarrow \infty$. In fig.

32, $ASK = \alpha$.

As θ increases beyond α , r becomes negative and for $\theta = \pi$, $r = -\frac{l}{e-1}$ which corresponds to A' in fig. 32. As θ increases further, r remaining negative until θ equals $2\pi - \alpha$, ASK' in the figure. When $\theta \rightarrow (2\pi - \alpha)$, $r \rightarrow -\infty$.

Finally, when θ increases from $(2\pi - \alpha)$ to 2π , the values of r remains positive.

The order in which the curve is described is shown below.

	(i)	(ii)	(iii)	(iv)
θ varies from	0 to α	α to π	π to $(2\pi - \alpha)$	$(2\pi - \alpha)$ to 2π
Portion described	AE	FA'	$A'G$	HA

Hence, the curve consists of two separate branches HAE and $FA'G$ and its shape is as in fig. 32.

Note 1. Since $\cos(-\theta) = \cos \theta$, it is enough to consider the variation in the value of θ from 0 to π . The curve from $\theta = \pi$ to $\theta = 2\pi$ can then be traced by symmetry about the initial line.

Note 2. The radius vector of any point on the further branch $FA'G$ of the hyperbola is negative. Thus, (fig. 32), if

a straight line through S meets the *nearer* branch in Q and the *further* branch in P , then the two points P and Q do not have the same vectorial angle.

The radius vector SP being negative, the vectorial angle of P is ASP' , where P' , is any point on PS produced. Hence, if the vectorial angle of Q be θ , that of P is $\theta - \pi$ or $\theta + \pi$,

i.e., if $SQ = \frac{l}{1 + e \cos \theta}$,

then $SP = -\frac{l}{1 + e \cos (\theta \pm \pi)} = -\frac{l}{1 - e \cos \theta}$(3)

Ex. 1. If PSP' and QSQ' be any two focal chords of a conic at right angles to one another, show that

(i) $\frac{1}{PP'} + \frac{1}{QQ'} = \text{constant}$. (Aligarh '58; Rajputana '58)

(ii) $\frac{1}{PS \cdot SP'} + \frac{1}{QS \cdot SQ'} = \text{constant}$. (Baroda '57)

Let the focus S be the pole and the equation of the conic

be $\frac{l}{r} = 1 + e \cos \theta$(1)

Let the vectorial angle of P be α , then the vectorial angles of P' , Q , Q' are respectively $\pi + \alpha$, $\frac{\pi}{2} + \alpha$, $\frac{3\pi}{2} + \alpha$. Hence, from (1), we have

$$\frac{l}{SP} = 1 + e \cos \alpha, \quad \frac{l}{SP'} = 1 + e \cos (\pi + \alpha) = 1 - e \cos \alpha,$$

$$\frac{l}{SQ} = 1 + e \cos \left(\frac{\pi}{2} + \alpha \right) = 1 - e \sin \alpha,$$

and $\frac{l}{SQ'} = 1 + e \cos \left(\frac{3\pi}{2} + \alpha \right) = 1 + e \sin \alpha;$

$$\therefore PP' = SP + SP' = \frac{l}{1 + e \cos \alpha} + \frac{l}{1 - e \cos \alpha} = \frac{2l}{1 - e^2 \cos^2 \alpha},$$

and $QQ' = SQ + QS' = \frac{2l}{1 - e^2 \sin^2 \alpha}.$

Hence,
$$\frac{1}{PP'} + \frac{1}{QQ'} = \frac{1 - e^2 \cos^2 \alpha}{2l} + \frac{1 - e^2 \sin^2 \alpha}{2l} = \frac{2 - e^2}{2l} = \text{constant},$$

and
$$\frac{1}{PS \cdot SP'} + \frac{1}{QS \cdot SQ'} = \frac{1 - e^2 \cos^2 \alpha}{l^2} + \frac{1 - e^2 \sin^2 \alpha}{l^2}$$

$$= \frac{2 - e^2}{l^2} = \text{constant}.$$

Ex. 2. *If two conics have a common focus, show that two of their common chords will pass through the point of intersection of their directrices.*
(Baroda '54; Agra '59)

Let the common focus be the pole, so that the equations of the two conics are

$$\frac{l}{r} = 1 + e \cos \theta, \quad \dots\dots (1)$$

and
$$\frac{l'}{r} = 1 + e' \cos (\theta - \gamma). \quad \dots\dots (2)$$

Transforming to rectangular cartesian coordinates and rationalizing, equations (1) and (2) become respectively

$$S \equiv (l - e x)^2 - (x^2 + y^2) = 0.$$

and
$$S' \equiv (l' - x e' \cos \gamma - y e' \sin \gamma)^2 - (x^2 + y^2) = 0.$$

Now $S - S' \equiv (l - e x)^2 - (l' - x e' \cos \gamma - y e' \sin \gamma)^2 = 0$ represents a pair of straight lines through the intersections of S and S' . Hence, changing back to polar coordinates, we see that two of the common chords of the conics (1) and (2) are given by the equation

$$\left(\frac{l}{r} - e \cos \theta\right)^2 - \left\{\frac{l'}{r} - e' \cos (\theta - \gamma)\right\}^2 = 0,$$

i.e.,
$$\frac{l}{r} - e \cos \theta = \pm \left\{\frac{l'}{r} - e' \cos (\theta - \gamma)\right\}.$$

Hence, the common chords pass through the intersections of $\frac{l}{r} = e \cos \theta$ and $\frac{l'}{r} = e' \cos (\theta - \gamma)$, which are respectively the directrices of the conics (1) and (2).

8.5. Equation of the chord joining the Points ' $\alpha - \beta$ ' and ' $\alpha + \beta$ ' on the Conic

$$\frac{1}{r} = 1 + e \cos \theta. \quad \dots\dots (1)$$

Let P, Q be the points on (1), whose vectorial angles are respectively $\alpha - \beta$ and $\alpha + \beta$. The equation of the chord PQ is of

the form

$$\frac{l}{r} = A \cos \theta + B \cos (\theta - \alpha), \quad \dots\dots(2)$$

Where A, B are constants; since on changing (2) to cartesians, the transformed equation is of the first degree in x and y .

If the line (2) passes through P and Q , we have

$$1 + e \cos (\alpha - \beta) = \frac{l}{SP} = A \cos (\alpha + \beta) + B \cos \beta, \quad \dots\dots(3)$$

$$\text{and } 1 + e \cos (\alpha + \beta) = \frac{l}{SQ} = A \cos (\alpha + \beta) + B \cos \beta, \quad \dots\dots(4)$$

where S is the focus of (1) which has been taken as the pole.

Solving (3) and (4), we get

$$A = e \text{ and } B = \sec \beta.$$

Hence, the equation of the chord PQ is

$$\frac{1}{r} = e \cos \theta + \sec \beta \cos (\theta - \alpha). \quad \dots\dots(5)$$

Cor. The equation of the chord joining ' $\alpha - \beta$ ' and ' $\alpha + \beta$ ' on the conic

$$\frac{l}{r} = 1 + e \cos (\theta - \gamma)$$

$$\text{is } \frac{l}{r} = e \cos (\theta - \gamma) + \sec \beta \cos (\theta - \alpha). \quad \dots\dots(6)$$

8.51. Equation of the Tangent at the Point ' α ' on the

Conic
$$\frac{1}{r} = 1 + e \cos \theta. \quad \dots\dots(1)$$

If $\beta \rightarrow 0$, the points P and Q of § 8.5 coincide, and the chord PQ in this limiting position becomes a tangent.

Hence, making $\beta \rightarrow 0$ in equation (5) of § 8.5, we see that the equation of the tangent at the point ' α ' on (1) is

$$\frac{1}{r} = e \cos \theta + \cos (\theta - \alpha) \quad \dots\dots(2)$$

Cor. The equation of the tangent at the point ' α ' on the conic

$$\frac{l}{r} = 1 + e \cos (\theta - \gamma)$$

$$\text{is } \frac{l}{r} = e \cos (\theta - \gamma) + \cos (\theta - \alpha). \quad \dots\dots(3)$$

8.6. Asymptotes of the Hyperbola

$$\frac{l}{r} = 1 + e \cos \theta. \quad \dots\dots(1)$$

Method I. The equation of the tangent at the point 'a' on (1) is (§ 8. 51)

$$\frac{l}{r} = e \cos \theta + \cos (\theta - a). \quad \dots\dots(2)$$

Now, an asymptote is the limiting position of a tangent when the point of contact tends to infinity (see § 6.4.) and the point 'a' tends to infinity (§ 8. 43), if

$$1 + e \cos a = 0. \quad \dots\dots(3)$$

Hence, the equation of the asymptotes is obtained by eliminating a from (2) and (3).

From (2), we have

$$\begin{aligned} \frac{l}{r} &= (e + \cos a) \cos \theta + \sin a \sin \theta \\ &= \left(e - \frac{1}{e}\right) \cos \theta \pm \sqrt{1 - \frac{1}{e^2}} \sin \theta, \text{ [from (3)]} \\ &= \frac{e^2 - 1}{e} \left[\cos \theta \pm \frac{\sin \theta}{\sqrt{e^2 - 1}} \right] \quad \dots\dots(4) \end{aligned}$$

which are the required equations of the asymptotes.

Cor. The equations of the asymptotes of the hyperbola

$$\frac{l}{r} = 1 + e \cos (\theta - \gamma)$$

are $\frac{l}{r} = \frac{e^2 - 1}{e} \left[\cos (\theta - \gamma) \pm \frac{\sin (\theta - \gamma)}{\sqrt{e^2 - 1}} \right] \quad \dots\dots(5)$

Method II. If p be the length of the perpendicular from the focus S (taken as the pole) upon any asymptote, then (fig. 32)

$$\begin{aligned} p &= CS \sin a, \text{ where } C \text{ is the centre and } \cos a = -\frac{1}{e} \\ &= a e \sqrt{1 - \frac{1}{e^2}}, \text{ where } a \text{ is the semi-transverse axis} \\ &= a \sqrt{e^2 - 1} = b, \text{ where } b \text{ is the semi-conjugate axis,} \end{aligned}$$

Also, the perpendicular from S on the asymptote parallel to SK (fig. 32) makes an angle $\alpha - \pi/2$ with the initial line ;
 \therefore the equation of this asymptote is

$$b = r \cos [\theta - (\alpha - \pi/2)] = -r \sin (\theta - \alpha). \quad \dots\dots (6)$$

Similarly, the equation of the other asymptote is

$$\begin{aligned} b &= r \cos [\theta - (2\pi - \alpha + \pi/2)] \\ &= r \sin (\theta + \alpha). \quad \dots\dots (7) \end{aligned}$$

Note 1. For another method see Ex. 4 below.

Note 2. As $l = a(e^2 - 1)$, equation (4) can be written as

$$a\sqrt{e^2 - 1} = r \left[\cos \theta \cdot \sqrt{1 - \frac{1}{e^2}} \pm \frac{1}{e} \sin \theta \right]$$

or $b = \pm r \sin (\theta \pm \alpha)$, since $\cos \alpha = -\frac{1}{e}$.

Thus (4) can be reduced to the form (6) or (7).

Ex. 1. QR , a chord of the conic $\frac{l}{r} = 1 - e \cos \theta$, subtends a constant angle 2α at its focus S , and SP , the bisector of the angle QSR , meets QR in P . Show that the locus of P is the conic

$$\frac{l \cos \alpha}{r} = 1 - e \cos \alpha \cos \theta \quad (\text{Agra '58; Allahabad '59})$$

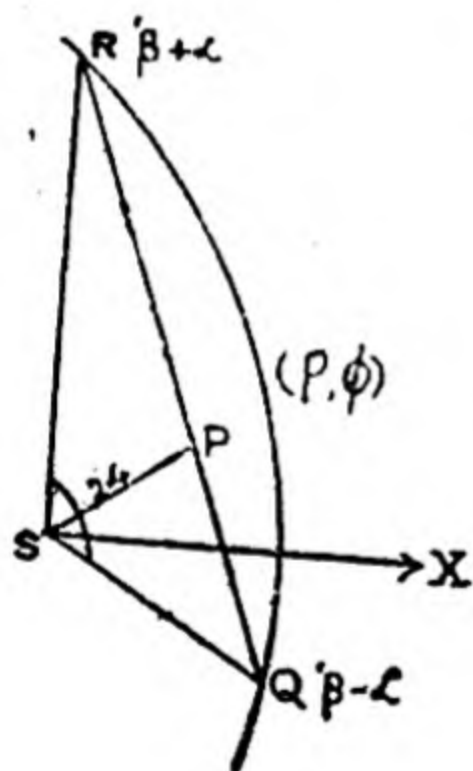


Fig. 33

The equation of the conic can be written as

$$\frac{l}{r} = 1 + e \cos (\theta - \pi), \quad \dots\dots (1)$$

Let the vectorial angles of the points Q, R be respectively $(\beta - \alpha)$ and $(\beta + \alpha)$, so that $QSR = 2\alpha$.

The equation of the chord QR is [see § 8.5 (6)]

$$\begin{aligned} \frac{l}{r} &= e \cos (\theta - \pi) + \sec \alpha \cos (\theta - \beta) \\ &= -e \cos \theta + \sec \alpha \cos (\theta - \beta). \quad \dots\dots (2) \end{aligned}$$

If the coordinates of P be (ρ, ϕ) , then

$$\phi = (\beta - \alpha) + \alpha = \beta, \quad \dots\dots (3)$$

and $\frac{l}{\rho} = -e \cos \phi + \sec \alpha \cos (\phi - \beta). \quad \dots\dots (4)$

Eliminating β from (3) and (4), we get

$$\frac{l}{\rho} = -e \cos \phi + \sec \alpha.$$

Hence, the locus of $P(\rho, \phi)$ is

$$\frac{l}{r} = -e \cos \theta + \sec \alpha,$$

i.e.

$$\frac{l \cos \alpha}{r} = 1 - e \cos \alpha \cos \theta.$$

Ex. 2. Show that the condition that the line

$$\frac{l}{r} = A \cos \theta + B \sin \theta$$

may touch the conic $\frac{l}{r} = 1 + e \cos \theta$

is $(A - e)^2 + B^2 = 1$. (Poona '58; Osmania '60)

If the line $\frac{l}{r} = A \cos \theta + B \sin \theta$ (1)

touches the conic at the point 'a', then equation (1) is identical with the equation of the tangent at 'a', i.e.,

$$\begin{aligned} \frac{l}{r} &= e \cos \theta + \cos (\theta - \alpha) \\ &= (e + \cos \alpha) \cos \theta + \sin \alpha \sin \theta. \end{aligned} \quad \text{.....(2)}$$

Comparing the coefficients in (1) and (2), we have

$$A = e + \cos \alpha, \quad B = \sin \alpha.$$

Eliminating α from these relations, we get

$$(A - e)^2 + B^2 = 1,$$

which is the required condition.

Ex. 3. Show that the equation of the locus of the point of intersection of two tangents to

$$\frac{l}{r} = 1 + e \cos \theta,$$

which are at right angles to one another,

is $r^2 (e^2 - 1) - 2elr \cos \theta + 2l^2 = 0$.

(Agra '54; Rajputana '57)

The equations of the tangents at any two points 'a' and 'b' on the given conic are

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha), \quad \dots\dots(1)$$

and

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \beta). \quad \dots\dots(2)$$

If the lines (1) and (2) meet in (r, θ) , then

$$\cos (\theta - \alpha) = \cos (\theta - \beta), \quad \text{i.e.,} \quad \theta - \alpha = \pm(\theta - \beta).$$

Since $\alpha \neq \beta$, it follows that $\theta - \alpha = -(\theta - \beta)$,

$$\text{i.e.,} \quad \theta = \frac{1}{2} (\alpha + \beta); \quad \dots\dots(3)$$

$$\text{and, therefore,} \quad \frac{l}{r} = e \cos \theta + \cos \frac{\alpha - \beta}{2}, \quad \dots\dots(4)$$

from either of the equations (1) and (2).

Now, the equations (1) and (2) can be written as

$$\left. \begin{aligned} \frac{l}{r} &= (e + \cos \alpha) \cos \theta + \sin \alpha \sin \theta. \\ \text{and} \quad \frac{l}{r} &= (e + \cos \beta) \cos \theta + \sin \beta \sin \theta; \end{aligned} \right\} \quad \dots\dots(5)$$

\therefore the two tangents are at right angles to one another, if

$$(e + \cos \alpha) (e + \cos \beta) + \sin \alpha \sin \beta = 0,$$

or

$$e^2 + e (\cos \alpha + \cos \beta) + \cos (\alpha - \beta) = 0,$$

$$\text{or} \quad e^2 + 2e \cos \frac{\alpha + \beta}{2} \cos \frac{\alpha - \beta}{2} + 2 \cos^2 \frac{\alpha - \beta}{2} - 1 = 0. \quad \dots\dots(6)$$

Hence, eliminating α, β from (3), (4) and (6), we get

$$e^2 + 2e \cos \theta \cdot \left(\frac{l}{r} - e \cos \theta \right) + 2 \left(\frac{l}{r} - e \cos \theta \right)^2 - 1 = 0,$$

$$\text{i.e.,} \quad r^2 (e^2 - 1) - 2ler \cos \theta + 2l^2 = 0, \quad \dots\dots(7)$$

which is the required locus.

Note 1. The equation (7) represents the *director circle* (also called *orthoptic circle*) of the given conic.

Note 2. When $e=1$, the director circle (7) reduces to
 $r \cos \theta = l,$

which is the equation of the directrix of the parabola

$$\frac{l}{r} = 1 + \cos \theta.$$

***Note 3.** This step can be easily written if we change equations (5) to cartesian.

Ex. 4. Show that the equation of the pair of tangents drawn to the conic is $\frac{l}{r} = 1 + e \cos \theta$ from the point (r', θ')

is $(S^2 - 1)(S'^2 - 1) = P^2$,

where $S \equiv \frac{l}{r} - e \cos \theta$, $S' \equiv \frac{l}{r'} - e \cos \theta'$,

and $P \equiv \left(\frac{l}{r} - e \cos \theta \right) \left(\frac{l}{r'} - e \cos \theta' \right) - \cos (\theta - \theta')$.

(Allahabad '60)

Hence, deduce the equation of the asymptotes.

(Poona '52; Aligarh '58)

The equation of the tangent at a point 'a' on the given conic is $\frac{l}{r} = e \cos \theta + \cos (\theta - a)$ (1)

This passes through (r', θ') , if

$$\frac{l}{r'} = e \cos \theta' + \cos (\theta' - a). \quad \dots\dots(2)$$

Eliminating a from (1) and (2), we get the equation of the pair of tangents drawn from (r', θ') to the conic.

From (1) and (2), we have

$$\begin{aligned} & \left\{ \left(\frac{l}{r} - e \cos \theta \right)^2 - 1 \right\} \left\{ \left(\frac{l}{r'} - e \cos \theta' \right)^2 - 1 \right\} \\ &= \{ \cos^2 (\theta - a) - 1 \} \{ \cos^2 (\theta' - a) - 1 \} \\ &= \sin^2 (\theta - a) \sin^2 (\theta' - a), \quad \dots\dots(3) \end{aligned}$$

$$\begin{aligned} \text{and } & \left(\frac{l}{r} - e \cos \theta \right) \left(\frac{l}{r'} - e \cos \theta' \right) - \cos (\theta - \theta') \\ &= \cos (\theta - a) \cos (\theta' - a) - \cos \{ (\theta - a) - (\theta' - a) \} \\ &= -\sin (\theta - a) \sin (\theta' - a). \quad \dots\dots(4) \end{aligned}$$

Hence, from (3) and (4), we get

$$\begin{aligned} & \left\{ \left(\frac{l}{r} - e \cos \theta \right)^2 - 1 \right\} \left\{ \left(\frac{l}{r'} - e \cos \theta' \right)^2 - 1 \right\} \\ &= \left\{ \left(\frac{l}{r} - e \cos \theta \right) \left(\frac{l}{r'} - e \cos \theta' \right) - \cos (\theta - \theta') \right\}^2, \quad \dots\dots(5) \end{aligned}$$

$$\text{i.e.,} \quad (S^2 - 1)(S'^2 - 1) = P^2. \quad \dots\dots(6)$$

Since the asymptotes of a conic are the pair of tangents

from its centre (§ 6.4), putting $r' = \frac{el}{e^2 - 1}$, the distance of the centre from the focus (see § 8.42) and $\theta' = 0$ in (5), we get the equation of the asymptotes as

$$\begin{aligned} & \left\{ \left(\frac{l}{r} - e \cos \theta \right)^2 - 1 \right\} \left(\frac{1}{e^2} - 1 \right) \\ &= \left\{ \left(\frac{l}{r} - e \cos \theta \right) \left(-\frac{1}{e} \right) - \cos \theta \right\}^2 \\ &= \frac{l^2}{e^2 r^2}, \end{aligned}$$

$$\begin{aligned} \text{or } & \left\{ \frac{l^2}{r^2} - 2 \frac{l}{r} \cdot e \cos \theta + e^2 \cos^2 \theta - (\cos^2 \theta + \sin^2 \theta) \right\} \\ & (e^2 - 1) + \frac{l^2}{r^2} = 0, \end{aligned}$$

$$\text{or } \frac{e^2 l^2}{r^2} - 2 \frac{el}{r} (e^2 - 1) \cos \theta + (e^2 - 1)^2 \cos^2 \theta = (e^2 - 1) \sin^2 \theta,$$

$$\text{or } \left\{ \frac{el}{r} - (e^2 - 1) \cos \theta \right\}^2 = (e^2 - 1) \sin^2 \theta,$$

which can be written as

$$\frac{el}{r} = (e^2 - 1) \cos \theta \pm \sqrt{e^2 - 1} \sin \theta,$$

$$\text{or } \frac{l}{r} = \frac{e^2 - 1}{e} \left\{ \cos \theta \pm \frac{\sin \theta}{\sqrt{e^2 - 1}} \right\}.$$

8.7. Equation of the Normal at the Point 'a' on the conic $\frac{l}{r} = 1 + e \cos \theta. \dots\dots (1)$

Since the equation of the tangent at the point 'a' is

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha),$$

the equation of the normal at 'a' is (§ 8.2, Note 2)

$$\begin{aligned} \frac{k}{r} &= e \cos \left(\theta - \frac{\pi}{2} \right) + \cos (\theta - \pi/2 - \alpha) \\ &= e \sin \theta + \sin (\theta - \alpha), \end{aligned} \dots\dots (2)$$

where k is such that (2) passes through the point 'a',

i.e., the point $\left[\frac{l}{1 + e \cos \alpha}, \alpha \right].$

$$\therefore \frac{k(1+e \cos \alpha)}{l} = e \sin \alpha,$$

$$\text{or } k = \frac{e \sin \alpha}{1+e \cos \alpha} l.$$

Hence, the equation of the normal at 'a' is

$$\frac{e \sin \alpha}{1+e \cos \alpha} \cdot \frac{l}{r} = e \sin \theta + \sin (\theta - \alpha). \quad \dots\dots(3)$$

8.8. Polar. To find the equation of the polar of the point (r_1, θ_1) w.r.t. the conic $\frac{l}{r} = 1 + e \cos \theta$(1)

In § 4.41, Part I, we have seen that the polar of a point P is the chord of contact of tangents (real or imaginary) drawn from P . We use this definition to find the polar of (r_1, θ_1) w.r.t. the conic (1).

Let ' $\alpha - \beta$ ' and ' $\alpha + \beta$ ' be the points of contact of tangents drawn from (r_1, θ_1) to the conic, then the equation of the chord of contact is (§ 8.5)

$$\frac{l}{r} = e \cos \theta + \sec \beta \cos (\theta - \alpha). \quad \dots\dots(2)$$

The equations of the tangents at ' $\alpha - \beta$ ' and ' $\alpha + \beta$ ' are respectively $\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha + \beta)$,(3)

and $\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha - \beta)$(4)

Since the lines (3) and (4) pass through (r_1, θ_1) , we have

$$\frac{l}{r_1} = e \cos \theta_1 + \cos (\theta_1 - \alpha + \beta), \quad \dots\dots(5)$$

and $\frac{l}{r_1} = e \cos \theta_1 + \cos (\theta_1 - \alpha - \beta)$,(6)

whence $\cos (\theta_1 - \alpha + \beta) = \cos (\theta_1 - \alpha - \beta)$,

i.e., $\theta_1 - \alpha + \beta = \pm (\theta_1 - \alpha - \beta)$.

Since $\beta \neq 0$, it follows that

$$\theta_1 - \alpha + \beta = -(\theta_1 - \alpha - \beta), \text{ i.e., } \theta_1 = \alpha. \quad \dots\dots(7)$$

Hence, from (5) or (6), we have

$$\frac{l}{r_1} = e \cos \theta_1 + \cos \beta,$$

i.e., $\cos \beta = \frac{l}{r} - e \cos \theta_1, \dots\dots(8)$

Eliminating α, β from (2), (7) and (8) we get

$$\left(\frac{l}{r} - e \cos \theta\right) \left(\frac{l}{r_1} - e \cos \theta\right) = \cos(\theta - \theta_1),$$

which is the required equation.

Ex. 1. If the normals at 'a', 'b', 'c' on $\frac{l}{r} = 1 + \cos \theta$ meet in the point (ρ, ϕ) , show that $2\phi = \alpha + \beta + \gamma$.

(Banaras '53; Allahabad '58)

The equation of the normal at the point ' μ ' on the given parabola is (§ 8.7)

$$\frac{\sin \mu}{1 + \cos \mu} \cdot \frac{l}{r} = \sin \theta + \sin(\theta - \mu),$$

or $\frac{l}{r} \tan \mu/2 = (1 + \cos \mu) \sin \theta - \sin \mu \cos \theta$

$$= \frac{2}{1 + \tan^2 \mu/2} \sin \theta - \frac{2 \tan \mu/2}{1 + \tan^2 \mu/2} \cos \theta,$$

or $lt^3 + (l + 2r \cos \theta)t - 2r \sin \theta = 0$, where $t = \tan \mu/2$.

If this normal passes through (ρ, ϕ) , then

$$lt^3 + (l + 2\rho \cos \phi)t - 2\rho \sin \phi = 0. \dots\dots(1)$$

The three roots of this cubic in t correspond to the three points 'a', 'b', 'c', the normals at which pass through (ρ, ϕ) .

Hence, from (1), we have

$$s_1 \equiv \Sigma \tan \alpha/2 = 0, \quad s_2 \equiv \Sigma \tan \alpha/2 \tan \beta/2 = \frac{l + 2\rho \cos \phi}{l},$$

and $s_3 \equiv \tan \alpha/2 \tan \beta/2 \tan \gamma/2 = \frac{2\rho \sin \phi}{l};$

$$\therefore \tan \left(\frac{\alpha + \beta + \gamma}{2} \right) = \frac{s_1 - s_3}{1 - s_2} = \frac{-2\rho \sin \phi}{l - (l + 2\rho \cos \phi)} = \tan \phi, \dots\dots(2)$$

i.e., $\frac{\alpha + \beta + \gamma}{2} = \phi,$

or $\alpha + \beta + \gamma = 2\phi.$

Note. The general solution of the equation (2) is $\alpha + \beta + \gamma = 2(n\pi + \phi)$, where n is an integer.

Ex. 2. Find the locus of the pole of a chord which subtends a constant angle 2α at a focus of a conic, distinguishing the cases

$$\cos \alpha \begin{matrix} > \\ = \\ < \end{matrix} e.$$

(Punjab '57; Agra '59)

Let P ' $\lambda - \alpha$ ' and Q ' $\lambda + \alpha$ ' be any two points on the conic

$$\frac{l}{r} = 1 + e \cos \theta, \quad \dots\dots(1)$$

so that the chord PQ subtends a constant angle 2α at the focus (taken as pole) of (1).

The equations of the tangents at P, Q are respectively

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \lambda + \alpha). \quad \dots\dots(2)$$

and $\frac{l}{r} = e \cos \theta + \cos (\theta - \lambda - \alpha). \quad \dots\dots(3)$

Since the pole of the chord PQ w. r. t. (1) is the point of intersection of the tangents at P, Q , the locus of the pole is obtained by eliminating λ between (2) and (3).

From (2) and (3), we have

$$\cos (\theta - \lambda + \alpha) = \cos (\theta - \lambda - \alpha),$$

i.e., $\theta - \lambda + \alpha = \pm (\theta - \lambda - \alpha),$

or $\theta = \lambda, \text{ as } \alpha \neq 0.$

Substituting for λ in (2) or (3), we get

$$\frac{l}{r} = e \cos \theta + \cos \alpha,$$

i.e., $\frac{l \sec \alpha}{r} = 1 + e \sec \alpha \cos \theta, \quad \dots\dots(4)$

which is the required equation of the locus of the pole.

The equation represents a fixed conic having the same focus and directrix as (1). The eccentricity of the conic (4) is $e \sec \alpha$ and the conic will be an ellipse, a parabola or a hyperbola according as $e \sec \alpha <, =, \text{ or } > 1,$
i.e., according as $\cos \alpha >, =, \text{ or } < 1.$

Note. 1. This problem can be stated as follows also :

If the chords of a conic subtend a constant angle at a focus, then

the tangents at the ends of any chord will meet on a fixed conic.

(Delhi '58; Allahabad '60)

Note. 2. It can be easily shown that the chord PQ , whose equation is $\frac{l}{r} = e \cos \theta + \sec \alpha \cos (\theta - \lambda)$, is a tangent, at the point ' λ ' on the conic

$$\frac{l \cos \alpha}{r} = 1 + e \cos \alpha \cdot \cos \theta.$$

(Delhi '58; Allahabad '60)

8.9. Miscellaneous Examples.

Ex. 1. If ϕ be the eccentric angle of the point (r, θ) on the ellipse $\frac{l}{r} = 1 + e \cos \theta$, prove that

$$\tan \frac{\theta}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{\phi}{2}.$$

Let the lengths of major and minor axes of the ellipse be respectively $2a$ and $2b$, then $l = \frac{b^2}{a} = a(1 - e^2)$.

Let C be the centre and S the focus of the ellipse, and draw PM , PN perpendiculars from $P(r, \theta)$ to the major axis and the minor axis respectively, then

$$PN = CS + SM,$$

i.e., $a \cos \phi = a e + SP \cos \theta$, where ϕ is the eccentric angle of P .

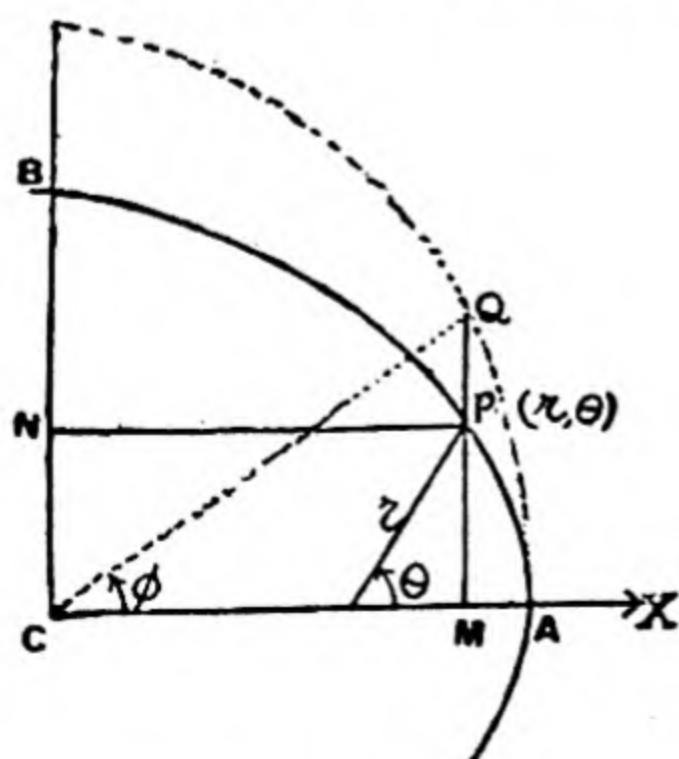


Fig. 34

$$\begin{aligned} \text{Now, } SP &= \frac{l}{1 + e \cos \theta} \\ &= \frac{a(1 - e^2)}{1 + e \cos \theta}; \end{aligned}$$

$$\begin{aligned} \therefore a \cos \phi &= a e + \frac{a(1 - e^2) \cos \theta}{1 + e \cos \theta} \\ &= a \cdot \frac{e + \cos \theta}{1 + e \cos \theta}. \end{aligned}$$

$$\text{Hence, } \cos \phi = \frac{e + \cos \theta}{1 + e \cos \theta},$$

or
$$\frac{1 - \cos \phi}{1 + \cos \phi} = \frac{(1 - e)(1 - \cos \theta)}{(1 + e)(1 + \cos \theta)},$$

or
$$\tan^2 \frac{\phi}{2} = \frac{1 - e}{1 + e} \tan^2 \frac{\theta}{2},$$

whence
$$\tan \frac{\phi}{2} = \sqrt{\frac{1 - e}{1 + e}} \tan \frac{\theta}{2}. \quad \dots\dots(1)$$

Note. The angles θ and ϕ are respectively known as the *True Anamoly* and *Eccentric Anamoly* in Astronomy.

The relation (1) above is of importance in the theory of elliptic orbits in Dynamics.

Ex. 2. Prove that the equation of the locus of the foot of the perpendicular from the focus of the conic $\frac{l}{r} = 1 + e \cos \theta$ on a tangent to it is.

$$r^2 (e^2 - 1) - 2ler \cos \theta + l^2 = 0.$$

Discuss the particular case when $e = 1$. (Allahabad 1955)

The equation of the tangent at any point 'a' on the conic is
$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha). \quad \dots\dots(1)$$

The equation of a st. line perpendicular to (1) and passing through the focus (i.e., the pole) is

$$\begin{aligned} 0 &= e \cos \left(\theta - \frac{\pi}{2} \right) + \cos \left(\theta - \frac{\pi}{2} - \alpha \right) \\ &= e \sin \theta + \sin (\theta - \alpha). \end{aligned} \quad \dots\dots(2)$$

Hence, the equation of the required locus is obtained by eliminating α between (1) and (2) and, therefore, is

$$\begin{aligned} \left(\frac{l}{r} - e \cos \theta \right)^2 + e^2 \sin^2 \theta &= \cos^2 (\theta - \alpha) + \sin^2 (\theta - \alpha) \\ &= 1, \end{aligned}$$

i.e.,
$$\frac{l^2}{r^2} - 2 \frac{l}{r} e \cos \theta + e^2 = 1,$$

or
$$r^2 (e^2 - 1) - 2ler \cos \theta + l^2 = 0. \quad \dots\dots(3)$$

When $e = 1$, the conic is a parabola and equation (3)

reduces to
$$\frac{l}{2} = r \cos \theta,$$

which represents the tangent at the vertex of the parabola; for, the distance of the vertex from the focus is $l/2$.

Note. The locus represented by (3) is called the *auxiliary circle* of the given conic.

Ex. 3. If a focal chord PSP' of an ellipse makes an angle α with the major axis, show that the angle between the tangents at P and P' is $\tan^{-1} \left(\frac{2e \sin \alpha}{1-e^2} \right)$. (Agra '55 S; Allahabad '58)

Let the equation of the ellipse referred to S as the pole and major axis as the initial line be

$$\frac{l}{r} = 1 + e \cos^1 \theta.$$

Since PSP' makes an angle α with the major axis, the vectorial angles of P, P' are respectively α and $\alpha + \pi$;

\therefore the equations of the tangents at P, P' are

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha), \quad \dots\dots(1)$$

and

$$\begin{aligned} \frac{l}{r} &= e \cos \theta + \cos (\theta - \alpha - \pi) \\ &= e \cos \theta - \cos (\theta - \alpha). \end{aligned} \quad \dots\dots(2)$$

The equation (1) can be written as

$$\frac{l}{r} = k \cos (\theta - \lambda),$$

where $k \cos \lambda = e + \cos \alpha$ and $k \sin \lambda = \sin \alpha$.

It follows, therefore, that the perpendicular to the line (1) makes an angle λ , i.e., $\tan^{-1} \frac{\sin \alpha}{e + \cos \alpha}$ with the initial line.

Similarly, the perpendicular to the line (2) makes an angle $\tan^{-1} \frac{-\sin \alpha}{e - \cos \alpha}$.

Hence, the angle between the tangents at P, P'

$$= \tan^{-1} \left(\frac{\sin \alpha}{e + \cos \alpha} \right) \curvearrowright \tan^{-1} \left(\frac{-\sin \alpha}{e - \cos \alpha} \right)$$

$$\begin{aligned}
&= \pm \tan^{-1} \left[\left(\frac{\sin \alpha}{e + \cos \alpha} + \frac{\sin \alpha}{e - \cos \alpha} \right) \middle/ \left(1 - \frac{\sin^2 \alpha}{e^2 - \cos^2 \alpha} \right) \right] \\
&= \pm \tan^{-1} \left(\frac{2e \sin \alpha}{e^2 - 1} \right) \quad \dots\dots(3)
\end{aligned}$$

Hence, if $0 < \alpha < \pi$, the acute value of the angle (3) is $\tan^{-1} \frac{2e \sin \alpha}{1 - e^2}$; for, $e < 1$.

Ex. 4. If the tangent from P subtend the fixed angle β at the focus S , prove that the locus of the middle point of SP is a conic of eccentricity $e \sec \beta$. (Lucknow 1956)

Let the equation of the conic referred to S as the pole be $\frac{l}{r} = 1 + e \cos \theta$, and let the vectorial angle of P be $\alpha - \beta$, then the vectorial angle of Q , the point of contact of the tangent from P , is $\alpha - \beta + \beta$, i.e., α , since $\angle PSQ = \beta$.

Now, the equation of the tangent at Q is

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha); \quad \dots\dots(1)$$

$$\therefore SP = \frac{l}{e \cos (\alpha - \beta) + \cos \beta}, \text{ since } P \text{ lies on (1).}$$

If $M(\rho, \phi)$ be the middle point of SP , then

$$2\rho = SP = \frac{l}{e \cos (\alpha - \beta) + \cos \beta}, \text{ and } \phi = \alpha - \beta$$

Eliminating α , we get

$$2\rho = \frac{l}{e \cos \phi + \cos \beta}$$

Hence, the equation of the locus of M is

$$2r = \frac{l}{e \cos \theta + \cos \beta},$$

$$\text{i.e., } \frac{l \sec \beta}{2r} = 1 + e \sec \beta \cos \theta, \quad \dots\dots(2)$$

which represents a conic of eccentricity $e \sec \beta$ and semi-latus rectum $\frac{l}{2} \sec \beta$.

Ex. 5. An ellipse and a parabola have a common focus S and intersect in two real points P and Q , of which P is the vertex of the

parabola. If e be the eccentricity of the ellipse and α the angle which SP makes with the major axis, prove that

$$\frac{SQ}{SP} = 1 + \frac{4e^2 \sin^2 \alpha}{(1 - e \cos \alpha)^2}.$$

Let the equations of the parabola and the ellipse be respectively

$$\frac{2a}{r} = 1 + \cos \theta, \quad \dots (1)$$

and
$$\frac{l}{r} = 1 + e \cos (\theta - \alpha). \quad \dots (2)$$

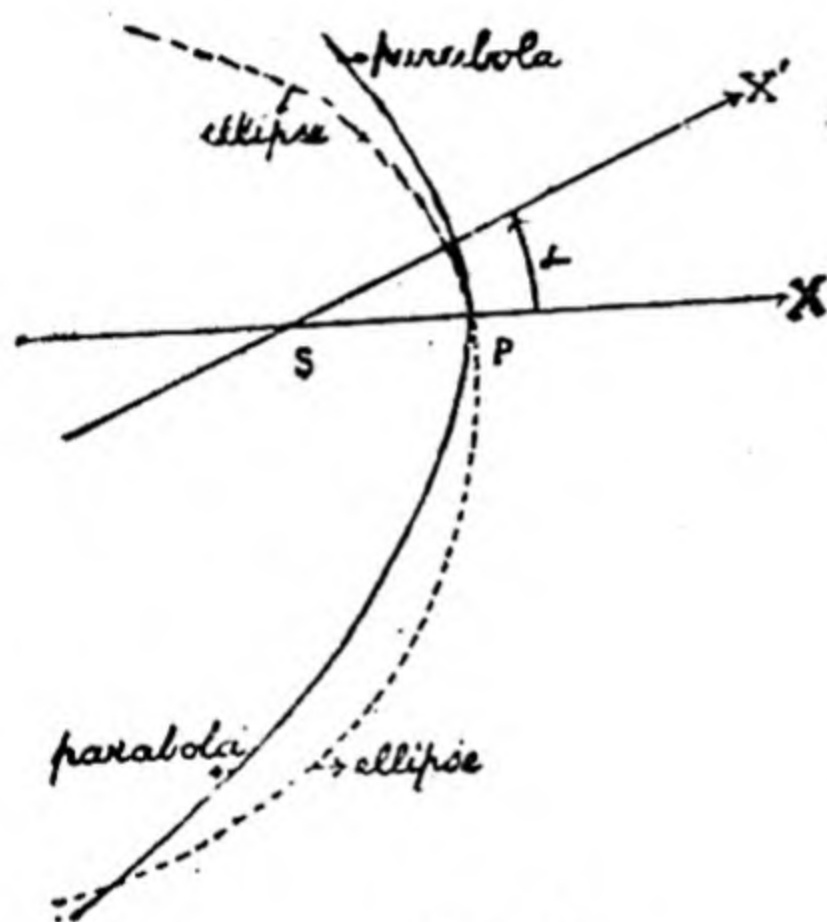


Fig. 35

Since (1) and (2) intersect in P ($\theta=0$), the vertex of (1), it follows that

$$SP = a = \frac{l}{1 + e \cos \alpha} \quad \dots (3)$$

To find the points of intersection, we eliminate r between (1) and (2), and get

$$\frac{2a}{l} = \frac{1 + \cos \theta}{1 + e \cos (\theta - \alpha)},$$

or $2[1 + e \cos (\theta - \alpha)] = (1 + \cos \theta)(1 + e \cos \alpha)$, from (3),
or, on simplifying,

$$(1 - \cos \theta)(1 - e \cos \alpha) = -2e \sin \theta \sin \alpha.$$

Now, the solution $\theta=0$ of this equation corresponds to the point P ; therefore, if $\theta \neq 0$, the point Q is given by

$$-\frac{2e \sin \alpha}{1 - e \cos \alpha} = \frac{1 - \cos \theta}{\sin \theta} = \tan \frac{\theta}{2}. \quad \dots (4)$$

Also, from (1), $r = a \sec^2 \frac{\theta}{2} = a (1 + \tan^2 \frac{\theta}{2})$;

$$\therefore SQ = a \left[1 + \frac{4e^2 \sin^2 \alpha}{(1 - e \cos \alpha)^2} \right], \text{ from (4).}$$

Hence,
$$\frac{SQ}{SP} = 1 + \frac{4e^2 \sin^2 \alpha}{(1 - e \cos \alpha)^2}, \quad \text{from (3).}$$

Exercise 8

1. Transform the following equations to polar coordinates :

$$\begin{array}{ll} \text{(i)} & y = mx + c. \\ \text{(ii)} & x^2 + y^2 - 2ax - 2by + c = 0. \\ \text{(iii)} & y^2 + 4ax = 4a^2. \\ \text{(iv)} & x^2 - y^2 = a^2. \end{array}$$

2. Transform the following equations to cartesian coordinates :

$$\begin{array}{ll} \text{(i)} & r \cos (\theta - \alpha) = p. \\ \text{(ii)} & A \cos (\theta - \alpha) = r. \\ \text{(iii)} & \frac{l}{r} = 1 + e \cos \theta. \\ \text{(iv)} & r^2 \sin 2\theta = 2c^2. \end{array}$$

3. Show that the area of a triangle whose vertices are

$$(2a, \theta), (a, \theta + \frac{\pi}{3}), (3a, \theta + \frac{2\pi}{3}) \text{ is } \frac{\sqrt{3}}{4} a^2.$$

4. Prove that any straight line passing through the point of intersection of the lines

$$\frac{1}{r} = A \cos \theta + B \sin \theta \text{ and } \frac{1}{r} = A' \cos \theta + B' \sin \theta$$

is $\frac{1+\lambda}{r} = (A + \lambda A') \cos \theta + (B + \lambda B') \sin \theta.$

Hence, obtain the equation of the line passing through the pole and the intersection-point of the given lines.

5. The polar coordinates of two points P_1, P_2 are $(r_1, \theta_1), (r_2, \theta_2)$. If the bisector of the angle P_1OP_2 meets P_1P_2 at P , prove that the polar coordinates of P are

$$\left[2r_1 r_2 \cos \frac{\theta_1 - \theta_2}{2} \mid (r_1 + r_2), \frac{1}{2} (\theta_1 + \theta_2) \right].$$

6. Show that the condition that the chord cut off by the curve $\frac{l}{r} = 1 + e \cos \theta$ from the line $\frac{1}{r} = a \cos \theta + b \sin \theta$ may subtend a right angle at the pole is

$$(la - e)^2 + l^2 b^2 = 2. \quad (\text{Aligarh 1959})$$

7. Find the condition that the line $\frac{1}{r} = a \cos \theta + b \sin \theta$ may touch the circle $r = 2c \cos \theta.$ (Punjab 1955)

8. A circle passes through the point (r_1, θ_1) and touches the initial line at a distance c from the pole. Show that its polar equation is

$$\frac{r^2 - 2cr \cos \theta + c^2}{r \sin \theta} = \frac{r_1^2 - 2cr_1 \cos \theta_1 + c^2}{r_1 \sin \theta_1}.$$

(Allahabad '58; Agra '60)

9. O is a fixed point and P any point on a given circle; OP is joined and on it a point Q is taken so that $OP.OQ = a$ constant quantity k . Prove that the locus of Q is a circle which becomes a straight line when O lies on the original circle.

(Allahabad 1942)

10. If from any point on the circum-circle of a triangle ABC , perpendiculars be drawn on the sides of the triangle, show that the feet of these perpendiculars will be collinear.

[Note. The line through the perpendicular is called the pedal line of the point O w. r. t. $\triangle ABC$.]

11. Find the centres of similitude of the circles $r^2 - 2ar \cos \theta + a^2 \cos^2 a = 0$, $r^2 - 2br \cos \theta + b^2 \cos^2 a = 0$ and show that the circle whose diameter is the line joining the centres of similitude is given by

$$(a+b)r = 2ab \cos \theta.$$

12. Prove that the equation of the tangent at (r_1, θ_1) to a circle whose centre is (c, a) is

$$r_1^2 = r_1 c \cos (\theta_1 - a) - cr \cos (a - \theta) + rr_1 \cos (\theta - \theta_1).$$

13. Show that the equations

$$\frac{l}{r} = 1 + e \cos \theta \text{ and } \frac{l}{r} = -1 + e \cos \theta$$

represent the same conic.

(Agra 1955)

[Hint :—Any point (r_1, θ_1) on the first conic can also be written as $(-r_1, \theta_1 + \pi)$.]

14. Show that in a conic the semi-latus rectum is the harmonic mean between the segments of a focal chord.

(Delhi 1958)

15. Prove that the perpendicular focal chords of a rectangular hyperbola are equal.

16. Prove that the locus of the mid-points of focal chords of a conic is a similar conic. *(Agra 1948)*

17. If PSQ , PHR be two chords of an ellipse through the foci S , H , then will $\frac{PS}{SQ} + \frac{PH}{HR}$ be independent of the position of P . *(Rajputana '51; Allahabad '60)*

18. A circle passes through the focus S of a conic and meets it in four points whose distances from S are r_1, r_2, r_3 and r_4 . Prove that

(i) $r_1 r_2 r_3 r_4 = \frac{d^2 l^2}{e^2}$, where $2l$ and e are the latus rectum and eccentricity of the conic and d is the diameter of the circle. *(Agra '57; Rajputana '57)*

$$(ii) \quad \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} + \frac{1}{r_4} = \frac{2}{l}.$$

(Agra '57; Allahabad '60)

19. If the tangent at any point on a conic whose focus is S meets the directrix in K , show that the angle PSK is a right angle. *(Nagpur 1954)*

20. If the tangents at any two points P and Q of a conic meet in a point T , and if the straight line PQ meets the directrix corresponding to S in a point K , prove that the angle KST is a right angle. *(Delhi '57; Vikram '61)*

21. If PQ is the chord of contact of the tangents drawn from a point T to a conic whose focus is S , prove that

(i) $ST^2 = SP \cdot SQ$, if the conic be a parabola; *(Sagar 1951)*

and (ii) $\frac{1}{SP \cdot SQ} - \frac{1}{ST^2} = \frac{1}{b^2} \sin^2 \frac{1}{2} PSQ$, if the conic be central with b as its semi-minor axis. *(Agra 1955)*

22. Prove that the two conics $\frac{l_1}{r} = 1 + e_1 \cos \theta$ and $\frac{l_2}{r} = 1 + e_2 \cos (\theta - \alpha)$ will touch one another, if

$$l_1^2 (1 - e_2^2) + l_2^2 (1 - e_1^2) = 2l_1 l_2 (1 - e_1 e_2 \cos \alpha).$$

(Agra '60; Vikram '60)

23. P, Q, R are three points on the conic

$$\frac{l}{r} = 1 + e \cos \theta.$$

the focus S being the pole; SP and SR meet the tangent at Q in M and N , so that $SM = SN = l$. Prove that PR touches the

conic $\frac{l}{r} = 1 + 2e \cos \theta$. (Poona '58; Rajputana '59)

24. Two conics have a common focus about which one of them is turned, prove that two of their common chords will touch conics having the same focus and whose eccentricity is the ratio of the eccentricities of the given conics. (Banaras '58)

25. If tangents be drawn at the points ' α ', ' β ', ' γ ' on the parabola $\frac{l}{r} = 1 + \cos \theta$ to form a triangle, show that the equation of the circum-circle of the triangle is

$$r = a \sec \frac{\alpha}{2} \sec \frac{\beta}{2} \sec \frac{\gamma}{2} \cos \left(\theta - \frac{\alpha + \beta + \gamma}{2} \right).$$

(Allahabad '55)

26. Two equal ellipses of eccentricity e are placed with their axes at right angles and they have one focus S in common; if PQ be a common tangent, show that the angle PSQ is equal

to $2 \sin^{-1} \frac{e}{\sqrt{2}}$. (Punjab '54; Agra '58)

27. A conic is described having the same focus and eccentricity as the conic $\frac{l}{r} = 1 + e \cos \theta$, and the two conics touch at the point $\theta = \alpha$; prove that the length of its latus rectum is $2l (1 - e^2) / (e^2 + 2e \cos \alpha + 1)$.

(Aligarh '53; Allahabad '56)

28. Show that the points ' 2α ' and ' 2β ' on the conic

$$\frac{l}{r} = 1 + e \cos \theta$$

are the extremities of a diameter, if

$$e + 1 = (e - 1) \tan \alpha \tan \beta.$$

[**Hint** :—The tangents at the two points are parallel.]

29. Two conics have the same focus and directrix. Show that any tangent from the outer curve to the inner one subtends a constant angle at the focus.

30. Show that the locus of the point of intersection of two perpendicular tangents, one drawn to each of the two parabolas with a common focus whose axes are neither coincident nor perpendicular, is a conic. (Agra 1952)

31. If the normals at three points of the parabola $r = a \operatorname{cosec}^2 \frac{\theta}{2}$, whose vectorial angles are α, β, γ meet in a point whose vectorial angle is δ , prove that

$$2\delta = \alpha + \beta + \gamma - \pi. \quad (\text{Rajputana 1955})$$

32. If the normals at ' α ', ' β ', ' γ ', ' δ ' on the conic $\frac{l}{r} = 1 + e \cos \theta$ meet in the point (ρ, ϕ) ; prove that

$$(i) \quad \tan \frac{\alpha}{2} \tan \frac{\beta}{2} \tan \frac{\gamma}{2} \tan \frac{\delta}{2} + \left(\frac{1+e}{1-e} \right)^2 = 0.$$

$$(ii) \quad \alpha + \beta + \gamma + \delta = (2n+1)\pi + 2\phi, \text{ where } n \text{ is an integer.}$$

33. If the normal at L , one of the extremities of the latus rectum of the conic $\frac{l}{r} = 1 + e \cos \theta$, meets the curve again at Q , show that

$$SQ = \frac{l(1+3e^2+e^4)}{1+e^2-e^4}, \quad (\text{Bombay '57; Agra '58})$$

34. Given the focus and directrix of a conic, show that the polar of a given point w. r. t. it passes through a fixed point. (Punjab 1953)

35. A chord PQ of a conic, whose eccentricity is e and semi-latus rectum l , subtends a right angle at the focus S . Prove that

$$\left(\frac{1}{SP} - \frac{1}{l} \right)^2 + \left(\frac{1}{SQ} - \frac{1}{l} \right)^2 = \frac{e^2}{l^2}.$$

36. If POP' be a chord of a conic through a fixed point, prove that $\tan \frac{1}{2} P'SO \cdot \tan \frac{1}{2} P'SO$ is constant, S being a focus of the conic.

37. If θ, θ' be the vectorial angles of any point on a given conic referred to the two foci, the initial line in both cases being the axis in the same sense, prove that the ratio

$$\tan \frac{\theta}{2} : \tan \frac{\theta'}{2} \text{ is constant.}$$

38. An ellipse and a hyperbola have the same focus S and intersect in four real points, two on each branch of the hyperbola; if r_1 and r_2 be the distances from S of the two points of intersection on the nearer branch, and r_3 and r_4 be those of the two points on the further branch, and if l and l' be the semi-lata recta of the two conics, prove that

$$(l+l') \left(\frac{1}{r_1} + \frac{1}{r_2} \right) + (l-l') \left(\frac{1}{r_3} - \frac{1}{r_4} \right) = 4.$$

Answers

1. (i) $r (\sin \theta - m \cos \theta) = c$; (ii) $r^2 - 2r (a \cos \theta + b \sin \theta) + c = 0$;

(iii) $\frac{2a}{r} = 1 + \cos \theta$; (iv) $r^2 \cos 2\theta = a^2$.

2. (i) $x \cos a + y \sin a = p$; (ii) $x^2 + y^2 - Ax \cos a - By \sin a = 0$;

(iii) $(l - ex)^2 = x^2 + y^2$; (iv) $xy = c^2$.

4. $\theta = \tan^{-1} \frac{A - A'}{B' - B}$.

7. $b^2 c^2 + 2ac = 1$.

11. $\left(\frac{2ab}{a+b}, 0 \right)$ and $(0, 0)$.

HINTS TO THE SOLUTIONS OF EXAMPLES

Exercise 1

1. Area of the triangle formed by the points ' t_1 ', ' t_2 ', and ' t_3 '.

$$= \frac{1}{2} \begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix} = -\frac{1}{2} a^2 (t_1 - t_2) (t_2 - t_3) (t_3 - t_1).$$

Area of the triangle formed by the tangents at these points

$$= \frac{1}{2} \begin{vmatrix} at_2t_3 & a(t_2 + t_3) & 1 \\ at_3t_1 & a(t_3 + t_1) & 1 \\ at_1t_2 & a(t_1 + t_2) & 1 \end{vmatrix} = \frac{1}{2} a^2 (t_1 - t_2) (t_2 - t_3) (t_3 - t_1).$$

2. Normals at the points ' t_2 ' and ' t_3 ' intersect in the point
 $\{ 2a + a(t_2^2 + t_2t_3 + t_3^2), a t_2t_3 (t_2 + t_3) \}$

Hence, the area of the triangle formed by the normals at ' t_1 ', ' t_2 ' and ' t_3 '

$$= \frac{1}{2} \begin{vmatrix} a(2 + t_2^2 + t_2t_3 + t_3^2) & a t_2t_3 (t_2 + t_3) & 1 \\ a(2 + t_3^2 + t_3t_1 + t_1^2) & a t_3t_1 (t_3 + t_1) & 1 \\ a(2 + t_1^2 + t_1t_2 + t_2^2) & a t_1t_2 (t_1 + t_2) & 1 \end{vmatrix}$$

$$= \frac{1}{2} a^2 (t_1 - t_2) (t_2 - t_3) (t_3 - t_1) (t_1 + t_2 + t_3)^2.$$

3. If the equation of a required st. line be

$$lx + my + 1 = 0, \quad \dots\dots(1)$$

then $l(x - x') + m(y - y') = 0. \quad \dots\dots(2)$

$$\text{Also, } \tan \theta = \pm \frac{-\frac{l}{m} + \frac{a}{b}}{1 + \frac{al}{bm}},$$

or $l(a \sin \theta \pm b \cos \theta) + m(b \sin \theta \mp a \cos \theta) = 0. \quad \dots\dots(3)$

Eliminating l, m from (1), (2) and (3), we get the required result.

4. Clearly

$$\lambda(a_2x + b_2y + c_2)(a_3x + b_3y + c_3) + \mu(a_3x + b_3y + c_3)(a_1x + b_1y + c_1) + \nu(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0, \quad \dots\dots(1)$$

will pass through the vertices of the triangle formed by the given lines. It will represent a circle, if

the coeff. of x^2 = the coeff. of y^2 and the coeff. of $xy = 0$,
 i.e., if $(a_2a_3 - b_2b_3)\lambda + (a_3a_1 - b_3b_1)\mu + (a_1a_2 - b_1b_2)\nu = 0$,(2)
 and $(a_2b_3 + a_3b_2)\lambda + (a_3b_1 + a_1b_3)\mu + (a_1b_2 + a_2b_1)\nu = 0$(3)

Eliminating λ, μ, ν between (1), (2) and (3), we get the required equation.

5. The three points are collinear, if

$$\begin{vmatrix} a^3 & a^2-3 & a-1 \\ b^3 & b^2-3 & b-1 \\ c^3 & c^2-3 & c-1 \end{vmatrix} = 0.$$

Simplifying, we get the required result.

6. Let the circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad \text{.....(1)}$$

$$\text{then } x_1x_2 + y_1y_2 + g(x_1 + x_2) + f(y_1 + y_2) + c = 0, \quad \text{.....(2)}$$

$$x_2x_3 + y_2y_3 + g(x_2 + x_3) + f(y_2 + y_3) + c = 0, \quad \text{.....(3)}$$

$$\text{and } x_3x_1 + y_3y_1 + g(x_3 + x_1) + f(y_3 + y_1) + c = 0. \quad \text{.....(4)}$$

Eliminating g, f, c from the equations (1), (2), (3) and (4), we get the required equation.

7. Putting $a_1 = \cos \alpha_1, b_1 = \sin \alpha_1, c_1 = -(a \sec \alpha_1 + b \sin \alpha_1)$ etc. in the equation of Ex. 6 above, we get the equation of the circum-circle as

$$\begin{vmatrix} 1 & & & \\ x \cos \alpha_1 + y \sin \alpha_1 - (a \sec \alpha_1 + b \sin \alpha_1) & \cos \alpha_1 & \sin \alpha_1 & \\ 1 & & & \\ x \cos \alpha_2 + y \sin \alpha_2 - (a \sec \alpha_2 + b \sin \alpha_2) & \cos \alpha_2 & \sin \alpha_2 & \\ 1 & & & \\ x \cos \alpha_3 + y \sin \alpha_3 - (a \sec \alpha_3 + b \sin \alpha_3) & \cos \alpha_3 & \sin \alpha_3 & \end{vmatrix} = 0.$$

It passes through $(0, b)$, if

$$\begin{vmatrix} -\frac{1}{a} \cos \alpha_1 & \cos \alpha_1 & \sin \alpha_1 \\ -\frac{1}{a} \cos \alpha_2 & \cos \alpha_2 & \sin \alpha_2 \\ -\frac{1}{a} \cos \alpha_3 & \cos \alpha_3 & \sin \alpha_3 \end{vmatrix} = 0, \text{ which is so since 2nd col.} \\ = -\frac{1}{a} \text{ (1st col.)}$$

8. Let the required equation be

$$x^2 + y^2 + 2gx + 2fy + k = 0, \quad \dots\dots(1)$$

then $-2a_1g - 2b_1f = c_1 + k$, or $k + c_1 + 2a_1g + 2b_1f = 0 \quad \dots\dots(2)$

and $-2a_2g - 2b_2f = c_2 + k$, or $k + c_2 + 2a_2g + 2b_2f = 0. \quad \dots\dots(3)$

Eliminating g, f from (1), (2) and (3), we get the result.

Exercise 2

1. If $y = mx$ be one of the st. lines given by the equation,
then $a + 3bm + 3cm^2 + dm^3 = 0. \quad \dots\dots(1)$

If the roots of (1) be m_1, m_2, m_3 then $m_1m_2m_3 = -a/d$.

Now, suppose that $m_1m_2 = -1$, then $m_3 = \frac{a}{d}$.

Substituting this value of m_3 in (1), we get the required condition.

2. Putting $y = x \tan \theta$, we get

$$m = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta} = \tan 3\theta,$$

$\therefore 3\theta = n\pi + \tan^{-1}m$. Hence etc.

3. If the given equation represents the four lines $y = m_r x$, $r = 1, 2, 3, 4$, then $m_1 m_2 m_3 m_4 = 1$. As two of the lines bisect the angles between the other two, they contain a rt. \angle , so that $m_1 m_2 = -1$, say. It follows that $m_3 m_4 = -1$.

Thus, the given equation represents two pairs of st. lines each containing a rt. \angle and one of these pairs bisects the angles between the other,

$\therefore ax^4 + bx^3y + cx^2y^2 + dxy^3 + ay^4$
 $\equiv -\frac{a}{\lambda} (x^2 + 2\lambda xy - y^2) (\lambda x^2 - 2xy - \lambda y^2), \text{ etc.}$

4. If the points of intersection of the given diagonal with the sides be (x_1, y_1) and (x_2, y_2) , then the equation of the other diagonal is $y = \frac{y_1 + y_2}{x_1 + x_2} x$.

Now x_1, x_2 are the roots of

$$ax^2 + 2hx \left(\frac{1 - lx}{m} \right) + b \left(\frac{1 - lx}{m} \right)^2 = 0, \text{ etc.}$$

5. If the perpendiculars are drawn from $P(x', y')$, then the distance between the feet of the perpendiculars is $OP \sin \theta$, where O is the origin and θ the angle between the lines;

$$\therefore 2k = \frac{\sqrt{(x'^2 + y'^2)} \cdot 2\sqrt{(h^2 - ab)}}{\sqrt{[(a-b)^2 + 4h^2]}}, \text{ etc.}$$

6. If the 3rd side be $lx + my = 1$, then the orthocentre (x', y') is given by

$$\begin{aligned} \frac{x'}{l} = \frac{y'}{m} &= \frac{a+b}{am^2 - 2hlm + bl^2} = \frac{bx'^2 - 2hx'y' + ay'^2}{a+b} \\ &= fx' + gy', \quad \therefore lf + mg = 1, \text{ etc.} \end{aligned}$$

7. (i) Let (x', y') , (x'', y'') be the points of intersection of $lx + my = 1$ and $ax^2 + 2hxy + by^2 = 0$, then

$$x_1 = \frac{1}{3} (x' + x'') = \frac{2}{3} \cdot \frac{bl - hm}{am^2 - 2hlm + bl^2},$$

and $y_1 = \frac{1}{3} (y' + y'') = \frac{2}{3} \cdot \frac{am - hl}{am^2 - 2hlm + bl^2};$

$$\therefore \frac{x'}{bl - hm} = \frac{y'}{am - hl} = \frac{2}{3} \cdot \frac{1}{am^2 - 2hlm + bl^2}. \quad \dots\dots(1)$$

(ii) If the 3rd side $lx + my = 1$ passes through (f, g) , then

$$lf + mg = 1. \quad \dots\dots(2)$$

From (1), we have

$$3(lx' + my') = 2, \quad \dots\dots(3)$$

and $(hx' + by')l - (ax' + hy')m = 0. \quad \dots\dots(4)$

Eliminating l, m from (2), (3) and (4), we get

$$\begin{vmatrix} f & g & 1 \\ 3x & 3y & 2 \\ hx' + by' & -(ax' + hy') & 0 \end{vmatrix} = 0, \text{ etc.}$$

8. If (x_1, y_1) be the point of intersection of the given lines, the equation of their angle-bisectors is

$$\frac{(x - x_1)^2 - (y - y_1)^2}{a - b} = \frac{(x - x_1)(y - y_1)}{h}$$

If α, β be the abscissae of the points in which these bisectors meet the x -axis, the area of the triangle $= \frac{1}{2} (\alpha - \beta)y_1$, etc.

9. Let $ax^2 + 2hxy + by^2 = 0$, represent the lines $y = m_1x$ and $y = m_2x$ and let ABC be the triangle whose sides AB, AC are respectively bisected at rt. \angle s by $y = m_1x$ and $y = m_2x$, then the slope of AB is $-\frac{1}{m_1}$ and that of AC is $-\frac{1}{m_2}$. Also, the origin O is the circum-centre of the $\triangle ABC$. If $OA = OB = OC = R$, the points A, B, C are respectively $(R \cos \alpha, R \sin \alpha)$, $(R \cos \beta, R \sin \beta)$ and $(R \cos \gamma, R \sin \gamma)$.

$$\text{Now, } -\frac{1}{m_1} = \text{slope of } AB = -\cot \frac{\alpha + \beta}{2}$$

$$\text{or } \tan \frac{\alpha + \beta}{2} = m_1. \quad \dots\dots(1)$$

$$\text{Similarly } \tan \frac{\gamma + \alpha}{2} = m_2; \quad \dots\dots(2)$$

$$\therefore \tan \left(\frac{\alpha + \beta}{2} + \frac{\gamma + \alpha}{2} \right) = \frac{m_1 + m_2}{1 - m_1 m_2} = \frac{2h}{a - b}. \quad \dots\dots(3)$$

Since BC passes through (f, g) .

$$\frac{g - R \sin \gamma}{f - R \cos \gamma} = -\cot \frac{\beta + \gamma}{2} \quad (\text{the slope of } BC),$$

$$\begin{aligned} \text{or } \frac{g - R \sin \gamma}{\cos \frac{\beta + \gamma}{2}} &= \frac{f - R \cos \gamma}{-\sin \frac{\beta + \gamma}{2}} \\ &= \frac{(g \cos \alpha + f \sin \alpha) - R \sin (\alpha + \gamma)}{\cos (\alpha + \frac{\beta + \gamma}{2})} \\ &= \frac{(g \sin \alpha - f \cos \alpha) + R \cos (\alpha + \gamma)}{\sin (\alpha + \frac{\beta + \gamma}{2})} \end{aligned}$$

$$\therefore \frac{g \cos \alpha + f \sin \alpha - R}{g \sin \alpha - f \cos \alpha + R} \cdot \frac{1 + m_2^2}{1 - m_2^2} = \frac{a - b}{2h}, \text{ from (2) and (3),}$$

etc.

10. The st. lines parallel to $S_1 \equiv ax^2 + 2hxy + by^2 = 0$ through (p, q) are given by

$$S_2 \equiv a(x - p)^2 + 2h(x - p)(y - q) + b(y - q)^2 = 0;$$

\therefore the required equation is $S_1 - S_2 = 0$, etc.

To find the area of the parallelogram, proceed as in solved Ex. 3, page 52, Part I.

11. p_1 , the perpendicular from (x_1, y_1) on $y - m_1 x = 0$ is $\frac{y_1 - m_1 x_1}{\sqrt{1 + m_1^2}}$, and p_2 , the perpendicular from (x_1, y_1) on $y - m_2 x = 0$ is $\frac{y_1 - m_2 x_1}{\sqrt{1 + m_2^2}}$, etc.

12. Any two circles belonging to the coaxal system determined by

$$S \equiv x^2 + y^2 + 2gx + c = 0 \text{ and } S' \equiv x^2 + y^2 + 2g'x + c = 0$$

are
$$x^2 + y^2 + 2 \frac{g + \lambda g'}{1 + \lambda} x + c = 0,$$

and
$$x^2 + y^2 + 2 \frac{g + \mu g'}{1 + \mu} x + c = 0.$$

These are orthogonal to each other, if

$$(g + \lambda g')(g + \mu g') = c(1 + \lambda)(1 + \mu).$$

If either λ or μ is given, it is always possible to find the other from the above equation. Hence etc.

For the second part, put $\lambda = \frac{a}{a'}$ and $\mu = -\frac{a}{a'}$.

13. Let the three coaxal circles be

$$x^2 + y^2 - 2g_r x + c = 0, \quad r = 1, 2, 3.$$

then their centres A, B, C are respectively $(g_1, 0)$, $(g_2, 0)$ and $(g_3, 0)$.

If tangents be drawn from (x_1, y_1) then

$$t_r^2 = x_1^2 + y_1^2 - 2g_r x_1 + c, \quad r = 1, 2, 3; \text{ etc.}$$

14. The polar of (x_1, y_1) w. r. t. $x^2 + y^2 + 2gx + c = 0$ is
$$xx_1 + yy_1 + g(x + x_1) + c = 0.$$

This is the same for every g , if

$$xx_1 + yy_1 + c \equiv k(x + x_1), \text{ where } k \text{ is a constant ;}$$

i.e. if $y_1 = 0$ and $x_1^2 = c$, etc.

15. The polars of $P(x_1, y_1)$ w. r. t. $x^2 + y^2 + 2gx + c = 0$ all pass through the fixed point $Q\left(-x_1, \frac{x_1^2 - c}{y_1}\right)$. The points P and Q are obviously equidistant from $x = 0$, the radical axis, etc.

16. The chord of intersection of the fixed circle

$$x^2 + y^2 + 2gx + 2fy + c = 0$$

with the coaxal system $x^2 + y^2 + 2ax + b = 0$ (where a is a parameter) is $2(g-a)x + 2fy + c - b = 0$

which always passes through the intersection of $2gx + 2fy + c - b = 0$ and $ax = 0$, etc.

17. The limiting points L, L' of the system

$$x^2 + y^2 + 2gx + c = 0 \quad \dots\dots(1)$$

are $(\sqrt{c}, 0)$ and $(-\sqrt{c}, 0)$ respectively. The centre C of (1) is $(-g, 0)$. Now $CL \cdot CL' = (g + \sqrt{c})(g - \sqrt{c}) = g^2 - c$, etc.

18. Let PQ be a common tangent of the circles

$$x^2 + y^2 + 2g_1x + c = 0, \quad \dots\dots(1)$$

$$\text{and} \quad x^2 + y^2 + 2g_2x + c = 0, \quad \dots\dots(2)$$

then R , the mid-point of PQ lies on $x = 0$, the radical axis of (1) and (2).

A circle with PQ as diameter will cut both (1) and (2) orthogonally and, therefore, has the equation.

$$x^2 + y^2 + 2fy - c = 0. \quad \dots\dots(3)$$

Hence, etc.

20. The limiting points of the system are

$$\left(-\frac{g}{1-\lambda_1}, -\frac{f\lambda_1}{1+\lambda_1}\right) \text{ and } \left(-\frac{g}{1+\lambda_2}, -\frac{f\lambda_2}{1+\lambda_2}\right),$$

where λ_1, λ_2 are the roots of

$$\left(\frac{g}{1+\lambda}\right)^2 + \left(\frac{f\lambda}{1+\lambda}\right)^2 - \frac{c+\lambda k}{1+\lambda} = 0,$$

$$\text{or} \quad (f^2 - k)\lambda^2 - (c + k)\lambda + g^2 - c = 0,$$

The two limiting points subtend a rt. \angle at $(0, 0)$, if

$$\frac{f\lambda_1}{g} \cdot \frac{f\lambda_2}{g} = -1, \text{ etc.}$$

21. The equation to the coaxal system is

$$(x-2)^2 + (y-1)^2 + \lambda [(x-1)^2 + (y-2)^2] = 0, \quad \dots\dots(1)$$

$$\text{or} \quad x^2 + y^2 - 2\frac{2+\lambda}{1+\lambda}x - 2\frac{1+2\lambda}{1+\lambda}y + 5 = 0. \quad \dots\dots(2)$$

The circle (2) can pass through the origin, only if $5 = 0$, which is absurd. Putting $x = 0, y = 0$ in (1), we get $5 + 5\lambda = 0$, i.e., $\lambda = -1$.

Substituting this value of λ in (1), we get $x-y=0$, etc.

22. Same as Ex. 2, page 16.

23. The square of the distance between the limiting points

$$=g^2 \left(\frac{1}{1+\lambda_2} - \frac{1}{1+\lambda_1} \right)^2 + f^2 \left(\frac{\lambda_2}{1+\lambda_2} - \frac{\lambda_1}{1+\lambda_1} \right)^2,$$

where λ_1 and λ_2 are the roots of

$$(f^2 - c') \lambda^2 - (c + c') \lambda + g^2 - c = 0, \text{ [see Ex. 20 above], Hence, etc.}$$

24. For different values of λ , each pair has the same radical axis

$$ax - by = 0.$$

Let the circle cutting orthogonally all the circles of the given system be $x^2 + y^2 + 2gx + 2fy + c = 0$,

then the condition $2a(1 + \frac{\lambda}{2})g + 2b(1 - \frac{\lambda}{2})f = c$ is true for all values of λ ; therefore the coefficient of λ and the constant term vanish separately.

Thus, $ag - bf = 0$ and $2ag + 2bf = c$, etc.

25. Let the required equation be

$$x^2 + y^2 + c^2 + 2\lambda x + k(x^2 + y^2 - c^2 + 2\mu y) = 0, \quad \dots\dots(1)$$

where k is a parameter, then the equation of the radical axis of (1) and the circle $x^2 + y^2 + c^2 + 2ax = 0$, $\dots\dots(2)$

$$\text{is} \quad (a - \lambda + ak)x - k\mu y + c^2k = 0. \quad \dots\dots(3)$$

The circles (1) and (2) touch each other, if

$$\frac{-(a - \lambda + ak)a + c^2k}{\sqrt{[(a - \lambda + ak)^2 + k^2\mu^2]}} = \sqrt{a^2 - c^2},$$

$$\text{i.e, if} \quad [a^2(c^2 + \mu^2) - c^2(c^2 + \mu^2)]k^2 = c^2(a - \lambda)^2. \quad \dots\dots(4)$$

Similarly, (1) and the circle $x^2 + y^2 + c^2 + 2bx = 0$,

touch each other, if

$$[b^2(c^2 + \mu^2) - c^2(c^2 + \mu^2)]k^2 = c^2(b - \lambda)^2. \quad \dots\dots(5)$$

From (4) and (5), by subtraction, we have

$$k^2 = \frac{c^2}{c^2 + \mu^2} \left(1 - \frac{2\lambda}{a+b} \right), \text{ etc.}$$

Exercise 3

1. Let the inclination of the latter axes to the former be θ , then the xy -term is removed if $\tan 2\theta = \frac{2 \cdot 1}{3-3} = \infty$, or $\theta = \pi/4$, etc.

2. The two equations are the equations of the same conic referred to rectangular axes having the same origin. Let the axes in the latter case make an angle θ with those in the former, then

$$a'x^2 + 2h'xy + b'y^2 = a(x \cos \theta - y \sin \theta)^2 + 2h(x \cos \theta - y \sin \theta)(x \sin \theta + y \cos \theta) + b(x \sin \theta + y \cos \theta)^2.$$

Now, equate the coefficients and eliminate θ .

3. By § 3.13, $\frac{1}{p} + \frac{1}{q} = a + b$ and $\frac{1}{pq} = ab - h^2$.

$$\begin{aligned} \text{Now, } \frac{X^2}{p-\lambda} &= \frac{Y^2}{q-\lambda} = \frac{qX^2 + pY^2 - \lambda(X^2 + Y^2)}{pq - (p+q)\lambda + \lambda^2} \\ &= \frac{X^2 p + Y^2 q - \frac{\lambda}{pq}(X^2 + Y^2)}{1 - \left(\frac{1}{p} + \frac{1}{q}\right)\lambda + \frac{\lambda^2}{pq}}, \text{ etc.} \end{aligned}$$

4. In the transformed equation, square and add the coefficients of X and Y .

6. The st. line becomes parallel to the new Y -axis.

7. A st. line through (h, k) is $y - k = m(x - h)$ (1)

It is perpendicular to the x -axis, if

$$1 + (m + 0) \cos \omega + 1 \times 0 = 0, \text{ i.e., } m = \frac{-1}{\cos \omega}, \text{ etc.}$$

8. Let M, N be the feet of the perpendiculars from $P(h, k)$ on the x -axis, y -axis respectively, then

$$PM = k \sin \omega, \text{ and } PN = h \sin \omega.$$

$$\text{Also, } \angle MPN = \pi - \omega,$$

$$\begin{aligned} \therefore MN^2 &= PM^2 + PN^2 - 2PM \cdot PN \cos \angle MPN \\ &= (h^2 + k^2 + 2hk \cos \omega) \sin^2 \omega. \end{aligned}$$

If PL be the length of the perpendicular from P upon MN , then

$$PL \cdot MN = 2\Delta PMN = PM \cdot PN \sin (\pi - \omega), \text{ etc.}$$

10. On changing to rectangular axes with the same origin and same x -axis, let $ax^2 + 2hxy + by^2$ be transformed into $AX^2 + 2HXY + BY^2$, where $X = x + y \cos \omega$, $Y = y \sin \omega$ and A is obviously equal to a .

The equation to the pair of bisectors now is

$$\frac{X^2 - Y^2}{A - B} = \frac{XY}{H},$$

or
$$\begin{vmatrix} AX + HY & HX + BY \\ X & Y \end{vmatrix} = 0.$$

Now, apply the method of invariants and transfer back to original coordinates.

Exercise 4

2. The line $lx + my = 1$ (1)

meets the x -axis in $A \left(\frac{1}{l}, 0 \right)$ and the y -axis in $B \left(0, \frac{1}{m} \right)$, say; then AB subtends a rt. \angle at (a, b) , if

$$\frac{b - \frac{1}{m}}{a - \frac{1}{l}} \times \frac{b - \frac{1}{m}}{a} = -1, \text{ i.e., if } \frac{a}{l} + \frac{b}{m} = a^2 + b^2. \quad \dots\dots(2)$$

Now show that (2) is also the condition that the line (1) should touch the given conic.

3. The slope of the tangent at (x_1, y_1) is $= -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}$;

\therefore the normal at (x_1, y_1) is

$$\frac{x - x_1}{ax_1 + hy_1 + g} = \frac{y - y_1}{hx_1 + by_1 + f}.$$

4. The chord of contact of tangents drawn from (x_1, y_1) is $axx_1 + h(xy_1 + x_1y) + byy_1 = 1$(1)

The equation of the st. lines joining the points of intersection of the line (1) and the given conic to the centre, i.e., the origin is

$$ax^2 + 2hxy + by^2 = [axx_1 + h(xy_1 + x_1y) + byy_1]^2, \quad \dots\dots(2)$$

Now write the condition that the lines (2) contain a rt. \angle , etc.

5. If the tangent at (x_1, y_1) on the given conic is parallel to $y=mx$, then $m = -\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f}$. Hence the equation of the locus of (x_1, y_1) etc.

6. The equation of the pair of tangents from (x_1, y_1) to the given conic is

$$(ax_1^2 + by_1^2 - 1)(ax^2 + by^2 - 1) = (axx_1 + byy_1 - 1)^2 \quad \dots\dots(1)$$

If the lines (1) meet the x -axis in the points $(x_1, 0)$ and $(x_2, 0)$, then x_1 and x_2 are the roots of $(ax_1^2 + by_1^2 - 1)(ax^2 - 1) = (axx_1 - 1)^2$. Now, put $x_1 \cup x_2 = 2c$, etc.

7. The polar of (x_1, y_1) w. r. t. the second conic is

$$axx_1 + h(xy_1 + x_1y) + byy_1 + g(x + x_1) + f(y + y_1) + c = 0.$$

This touches the first conic, if

$$a^2(ax_1 + hy_1 + g)^2 + b^2(hx_1 + by_1 + f)^2 = (gx_1 + fy_1 + c)^2.$$

Hence, etc.

8. The polar of (x_1, y_1) w. r. t. the conic $x^2 + y^2 = e^2(x - a)^2$ is $xx_1 + yy_1 - e^2(x - a)(x_1 - a) = 0$, which always passes through the point of intersection of $xx_1 + yy_1 = 0$, and $e^2(x - a)(x_1 - a) = 0$, etc.

9. If a chord of the given conic with its middle point as (x_1, y_1) be parallel to the given line, then

$$-\frac{ax_1 + hy_1 + g}{hx_1 + by_1 + f} = \frac{l}{m}, \quad \dots\dots(1)$$

whence we get the locus of (x_1, y_1) .

If the ends of the chord $lx + my + n = 0$ are equidistant from (x_0, y_0) , then $\frac{y_0 - y_1}{x_0 - x_1} \times \frac{-l}{m} = -1$, i.e., $\frac{l}{m} = \frac{x_1 - x_0}{y_1 - y_0}$.

Hence, etc.

10. If the conjugate diameters of the conic be parallel to $y=mx$ and $y=m'x$, then

$$Ax^2 + 2Hxy + By^2 \equiv B(y - mx)(y - m'x), \quad \dots\dots(1)$$

$$\text{where} \quad a + h(m + m') + bmm' = 0. \quad \dots\dots(2)$$

Substituting the values of $(m + m')$ and mm' from (1) in the equation (2), we have

$$a + h \cdot \frac{-2H}{B} + b \cdot \frac{A}{B} = 0,$$

or $aB - 2hH + bA = 0$, which is the required condition.

11. The equations to two concentric conics can be written as

$$ax^2 + 2hxy + by^2 = 1 \text{ and } a'x^2 + 2h'xy + b'y^2 = 1.$$

The diameters $Ax^2 + 2Hxy + By^2 = 0$ are conjugate w. r. t. both, if $bA + aB - 2hH = 0$,(1)

and $b'A + a'B - 2h'H = 0$(2)

From (1) and (2), we get only one solution

$$\frac{A}{2(ha' - h'a)} = \frac{B}{2(bh' - b'h)} = \frac{H}{ba' - b'a}.$$

Hence, the only pair of common conjugate diameters of the two conics is

$$(ha' - h'a)x^2 + (ba' - b'a)xy + (bh' - b'h)y^2 = 0.$$

12. The equations of the pairs of tangents drawn from $(c, 0)$ and $(-c, 0)$ to the given conic are respectively

$$x^2 + 2hcxy - (abc + h^2c - 2b)cy^2 - 2cx + 2hc^2y - c^2 = 0, \quad \dots\dots(1)$$

$$\text{and } x^2 - 2hcxy - (abc + h^2c + 2b)cy^2 + 2cx + 2hc^2y - c^2 = 0. \quad \dots\dots(2)$$

Subtracting (2) from (1), we get $by^2 + hxy = x$, which is the equation of the conic through the four points of intersection of (1) and (2).

Proceed similarly for tangents drawn from points on the y -axis equidistant from the origin.

Exercise 5

1. If the feet of the normals from (h, k) be

$$(am_r^2, -2am_r), r = 1, 2, 3,$$

then m_1, m_2, m_3 are the roots of $am^3 + m(2a - h) + k = 0$;

$$\therefore \Sigma m_1 = 0, \Sigma m_1 m_2 = (2a - h)/a, m_1 m_2 m_3 = -k/a. \quad \dots\dots(1)$$

(i) If the normal corresponding to, say, $r = 2$ bisects the angle between the other two, then

$$\frac{m_1 + m_3}{1 - m_1 m_3} = \frac{2m_2}{1 - m_2^2}. \quad \dots\dots(2)$$

Now eliminate the m 's between the relations (1) and (2), etc.

$$(ii) \quad \frac{1}{2} \begin{vmatrix} am_1^2 & -2am_1 & 1 \\ am_2^2 & -2am_2 & 1 \\ am_3^2 & -2am_3 & 1 \end{vmatrix} = \text{constant.}$$

$$\text{i.e.,} \quad (m_1 - m_2)(m_2 - m_3)(m_3 - m_1) = \text{constant.} \quad \dots\dots(3)$$

Now eliminate the m 's between the relations (1) and (3), etc.

2. Let O be the point (h, k) and P, Q, R be respectively the points $(am_r^2, -2am_r)$, $r=1, 2, 3$; then

$$\Sigma m_1 = 0, \quad \Sigma m_1 m_2 = (2a - h)/a, \quad m_1 m_2 m_3 = -k/a. \quad \dots\dots(1)$$

$$\text{Now } SP \cdot SQ \cdot SR = a^3 (1 + m_1^2)(1 + m_2^2)(1 + m_3^2).$$

Using (1) show that the expression on the right is $a \cdot OS^2$.

3. Let $lx + my = 1$ meet the parabola in the points $P(am_1^2, -2am_1)$ and $Q(am_2^2, -2am_2)$, then $\frac{-2a(m_1 - m_2)}{a(m_1^2 - m_2^2)} = -\frac{l}{m}$
or $m_1 + m_2 = \frac{2m}{l}$.

If the normals at P and Q meet in (h, k) , then the foot of 3rd normal from (h, k) is $(am_3^2, -2am_3)$, where $\Sigma m_1 = 0$, etc.

$$\text{Hence, } \frac{2m}{l} = m_1 + m_2 = -m_3, \text{ etc.}$$

4. If P, Q, R be respectively the points $(am_r^2, -2am_r)$, $r=1, 2, 3$, then $\Sigma m_1 = 0$, $\Sigma m_1 m_2 = (2a - h)/a$, $m_1 m_2 m_3 = -k/a$.
 $\dots\dots(1)$

The st. line through P and perpendicular to QR is

$$y + 2am_1 = \frac{a(m_2^2 - m_3^2)}{2a(m_2 - m_3)} \quad x = -\frac{m_1}{2}(x - am_1^2), \text{ by (1)}$$

$$\text{or} \quad m_1 x + 2y = (h - 6a)m_1 - k. \quad \dots\dots(2)$$

Similarly, the st. line through Q and perpendicular to RP is

$$m_2 x + 2y = (h - 6a)m_2 - k. \quad \dots\dots(3)$$

Now find the point of intersection of (2) and (3), etc.

5. If the feet of the normals from (h, k) are the points t_r , $r=1, 2, 3$,

then t_1, t_2, t_3 are the roots of $at^3 + (2a - h)t - k = 0$, $\dots\dots(1)$

The tangents at ' t_1 ' and ' t_2 ' intersect in the point given by

$$\begin{cases} x = at_1t_2 = -\frac{k}{t_3}, \\ y = a(t_1 + t_2) = -at_3. \end{cases} \begin{cases} \because \text{ from (1), } \Sigma t_1 = 0, \\ \text{and } t_1t_2t_3 = k/a. \end{cases}$$

Since t_3 is a root of (1), the result follows.

7. Since $\alpha + \beta + \gamma + \delta = (2n + 1)\pi$, n an integer,
 $\therefore \cos(\alpha + \beta) = -\cos(\gamma + \delta)$ etc.

8. The normal at ' ϕ ' passes through (h, k) , if

$$\frac{ah}{\cos \phi} - \frac{bk}{\sin \phi} = a^2 - b^2, \quad \dots\dots(1)$$

$$\text{i.e.,} \quad [ah - (a^2 - b^2) \cos \phi]^2 (1 - \cos^2 \phi) = b^2 k^2 \cos^2 \phi;$$

$$\therefore \Sigma \cos a = \frac{2ah}{a^2 - b^2} \quad \text{and} \quad \Sigma \sec a = \frac{2(a^2 - b^2)}{ah}.$$

$$\text{Hence, } \Sigma \cos a \cdot \Sigma \sec a = 4. \quad \dots\dots(2)$$

Similarly, writing (1) as a biquadratic in $\sin \phi$, we get

$$\Sigma \sin a \cdot \Sigma \operatorname{cosec} a = 4. \quad \dots\dots(3)$$

Changing $a \cos a$ to x_1 , $b \sin a$ to y_1 etc., we get

$$\Sigma x_1 \cdot \Sigma \frac{1}{x_1} = \Sigma y_1 \cdot \Sigma \frac{1}{y_1} = 4.$$

9. The mean position of the four points ' ϕ_r ', $r = 1, 2, 3, 4$
 is $\left(\frac{a}{4} \Sigma \cos \phi_1, -\frac{b}{4} \Sigma \sin \phi_1 \right).$

As shown in Ex. 8 above, $\Sigma \cos \phi_1 = \frac{2aa}{a^2 - b^2}$, etc.

10. Normal at (x_1, y_1) is

$$\frac{x - x_1}{x_1 a^2} = \frac{y - y_1}{y_1 b^2},$$

$$\text{or} \quad b^2 y x_1 - a^2 x y_1 + (a^2 - b^2) x_1 y_1 = 0.$$

Write similarly the equations of the normals at (x_2, y_2) ; (x_3, y_3) and eliminate $b^2 y$, $a^2 x$, $(a^2 - b^2)$ from the three equations.

11. The abscissae of the points of intersection of the given ellipse and the circle $x^2 + y^2 + 2gx + 2fy + d = 0$ are given by $\{ (a^2 - b^2) x^2 + 2ga^2 x + a^2 b^2 + a^2 d \}^2 = 4af^2 b^2 (a^2 - x^2),$

whence $x_1 + x_2 + x_3 + x_4 = -\frac{4ga^2}{a^2 - b^2}$.

But $x_1 + x_2 + x_3 = 3h$; $\therefore x_4 = -\frac{4ga^2}{a^2 - b^2} - 3h$.

Similarly, $y_4 = -\frac{4fb^2}{b^2 - a^2} - 3k$.

Now (x_4, y_4) lies on the ellipse etc.

12. In Ex. 11 above, take $x_1 = a$, $x_2 + x_3 = 0$,

$$x_4 = -\frac{4ga^2}{a^2 - b^2} - a;$$

similarly, $y_4 = -\frac{4fb^2}{b^2 - a^2} - \beta$.

Now (x_4, y_4) lies on the ellipse, etc.

13. If 'a', 'β', 'γ' be the vertices of the triangle, its centroid is given by $3x = a \Sigma \cos a$, $3y = b \Sigma \sin a$.

Now in an equilateral triangle the centroid coincides with the circum-centre, etc.

14. Let the conjugate diameters of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ be $y = m_1x$ and $y = m_2x$, where $m_1m_2 = -b^2/a^2$ (1)

The diameter $y = m_1x$ meets the directrix $x = \frac{a}{e}$ in $(\frac{a}{e}, \frac{m_1a}{e})$ and the perpendicular from this point on the opposite side is

$$m_2y + x = \frac{a}{e} (m_1m_2 + 1) = \frac{a}{e} (1 - \frac{b^2}{a^2}) = ae. \quad \dots\dots (2)$$

Now the orthocentre is the point of intersection of (2) and $y = 0$.

15. The mid-point of P 'φ' and D 'φ + π/2' is given by

$$2x = a (\cos \phi - \sin \phi), \quad 2y = b (\sin \phi + \cos \phi),$$

Eliminate φ.

16. If P and Q are respectively the points 'φ' and 'φ + π/2' the circles on CP, CQ as diameters are.

$$x(x - a \cos \phi) + y(y - b \sin \phi) = 0,$$

or $x^2 + y^2 = ax \cos \phi + by \sin \phi$;
 and $x(x + a \sin \phi) + y(y - b \cos \phi) = 0$,
 or $x^2 + y^2 = -ax \sin \phi + by \cos \phi$.

Now eliminate ϕ .

17. Let ' ϕ ', ' $\phi + \pi$ ' be the extremities of the diameter PCP' and ' $\phi + \pi/2$ ', ' $\phi + 3\pi/2$ ' the extremities of the conjugate diameter DCD' . Take any point ' θ ' on the ellipse and find out $\tan \lambda$, $\tan \lambda'$ etc.

18. The tangent at ' ϕ ' to the ellipse is

$$\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = 1. \quad \dots\dots(1)$$

The director circle is $x^2 + y^2 = a^2 + b^2$. $\dots\dots(2)$

Now write the equation of the st. lines joining the origin to the points of intersection of (1) and (2) and show that the product of their slopes is $-b^2/a^2$.

19. If $Ax^2 + 2Hxy + By^2 \equiv B(y - m_1 x)(y - m_2 x)$,
 then $m_1 m_2 = \frac{A}{B} = -\frac{b^2}{a^2}$.

20. The parallel tangents $\frac{x}{a} \cos \phi + \frac{y}{b} \sin \phi = \pm 1$ are
 met by $\frac{x}{a} \cos \theta + \frac{y}{b} \sin \theta = 1$ in points given by

and $\frac{x/a}{\sin \theta - \sin \phi} = \frac{y/b}{\cos \theta - \cos \phi} = \frac{1}{-\sin(\theta - \phi)}$,
 $\frac{x/a}{\sin \theta + \sin \phi} = \frac{y/b}{-(\cos \theta + \cos \phi)} = \frac{1}{-\sin(\theta - \phi)}$.

Now show that the product of the slopes of the st. lines joining these points to the origin is $-b^2/a^2$. Hence, etc.

21. If CD be the semi-diameter conjugate to CP , then
 $r^2 + CD^2 = a^2 + b^2$, and $p \cdot CD = ab$.

Eliminate CD .

22. The conjugate diameters $y = \lambda x$ and $y = -\frac{b^2}{a^2 \lambda} x$ meet
 the given st. line in

$$P \left(\frac{\lambda}{l + m\lambda}, \frac{\lambda}{l + m\lambda} \right) \text{ and } Q \left(\frac{a^2 \lambda}{a^2 l \lambda - b^2 m}, \frac{-b^2}{a^2 l \lambda - b^2 m} \right).$$

The st. lines through P and Q perpendicular to the corresponding diameters are respectively

$$(my-1)\lambda^2 + (mx+by)\lambda + (lx-1) = 0, \quad \dots\dots(1)$$

and $a^4(lx-1)\lambda^2 - a^2b^2(mx+ly)\lambda + b^4(my-1) = 0. \quad \dots\dots(2)$

Now eliminate λ between (1) and (2).

23. If the points ' ϕ ' and ' $\phi + \pi/2$ ' of the first ellipse lie on the second ellipse, then

$$\frac{a^2 \cos^2 \theta}{a^2} + \frac{b^2 \sin^2 \theta}{\beta^2} = 1, \text{ and } \frac{a^2 \sin^2 \theta}{a^2} + \frac{b^2 \cos^2 \theta}{\beta^2} = 1.$$

Now eliminate θ .

24. Let P and Q be respectively the points ' θ ' and ' ϕ ',

then the normal at P is $\frac{ax}{\cos \theta} - \frac{by}{\sin \theta} = a^2 - b^2, \quad \dots\dots(1)$

and the semi-diameter CQ is $y = \frac{b \sin \phi}{a \cos \phi} x. \quad \dots\dots(2)$

Now show that if (1) be conjugate to (2) the condition remains the same on interchanging θ and ϕ .

25. If chords are drawn parallel to the major axis of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ through $P(a \cos \theta, b \sin \theta)$ and $D(-a \sin \theta, b \cos \theta)$, then P' and D' are respectively the points $(-a \cos \theta, b \sin \theta)$ and $(a \sin \theta, b \cos \theta)$. Hence etc.

26. If P, D are respectively the points ' ϕ ' and ' $\phi + \pi/2$ ', then T and T' will be respectively the points $(a \sec \phi, 0)$, and $(a, b \sec \phi)$. Hence etc.

Exercise 6

1. Apply the definition.

2. Here $a^2=9$, $b^2=4$ etc.

3. To find the locus of the point of intersection, eliminate m .

4. Compare the equation of the given line with that of the tangent at ' ϕ '.

5. The perpendicular from the centre upon the tangent

$$\frac{x}{a} - \frac{y}{b} \sin \phi = \cos \phi \quad \text{is} \quad y = \frac{-a}{b} x \sin \phi.$$

Eliminate ϕ to get the locus of Q .

Similarly, to get the locus of R , eliminate ϕ between

$$ax \sin \phi + by = (a^2 + b^2) \tan \phi \quad \text{and} \quad y = \frac{b}{a \sin \phi} x,$$

6. The chord through ' θ ' and ' ϕ ' is

$$\frac{x}{a} \cos \frac{1}{2} (\theta - \phi) - \frac{y}{b} \sin \frac{1}{2} (\theta + \phi) = \cos \frac{1}{2} (\theta + \phi).$$

It passes through $(ae, 0)$, if

$$e \cos \frac{1}{2} (\theta - \phi) = \cos \frac{1}{2} (\theta + \phi), \text{ etc.}$$

7. The normal at ' ϕ ' meets x -axis in $M \left(\frac{a^2 + b^2}{a} \sec \phi, \right.$

0) and y -axis in $N \left(0, \frac{a^2 + b^2}{b} \tan \phi \right)$. The equations to MP

and NP are respectively $x = \frac{a^2 + b^2}{a} \sec \phi$, $y = \frac{a^2 + b^2}{b} \tan \phi$.

Eliminate ϕ to get the locus of P .

8. If (x_1, y_1) be the pole of the normal

$$ax \cos \phi + by \cot \phi = a^2 + b^2, \quad \dots\dots(1)$$

$$\text{then (1) is the same as } \frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1. \quad \dots\dots(2)$$

Now eliminate ϕ between (1) and (2).

9. The chord of contact of tangents drawn from $(a \cos \theta, a \sin \theta)$ to the hyperbola is $x \cos \theta - y \sin \theta = a$, $\dots\dots(1)$

and the chord of the hyperbola having (x_1, y_1) as its mid-point

$$\text{is} \quad xx_1 - yy_1 = x_1^2 - y_1^2. \quad \dots\dots(2)$$

Now compare (1) and (2) and eliminate θ .

10. Any tangent to the hyperbola is

$$x \sec \theta - y \tan \theta = a. \quad \dots\dots(1)$$

If (x_1, y_1) be the pole of (1) w. r. t. the parabola, then (1)

$$\text{is identical with} \quad yy_1 = 2a(x + x_1). \quad \dots\dots(2)$$

Now compare (1) and (2) and eliminate θ .

11. The polar of $(a \cos \theta, b \sin \theta)$ w. r. t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

i. e., $\frac{x \cos \theta}{a} - \frac{y \sin \theta}{b} = 1$ obviously touches the ellipse at the point $(a \cos \theta, -b \sin \theta)$.

12. Proceed as in § 5.4 and solved Ex. on page 54.

13. Proceed as in Ex. 8. page 63.

14. Find c such that $(x+2y+3)(3x+4y+5)+c=0$ passes through $(1, -1)$, etc.

15. Proceed as in solved Ex. 1, page 85.

16. The hyperbolas $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \dots\dots(1)$

and $\frac{x^2}{a^2} - \frac{y^2}{b^2} = k \dots\dots(2)$

have the same asymptotes.

From any point $(a \sec \theta, b \tan \theta)$ on (1) the chord of contact of tangents drawn to (2) is $\frac{x}{a} \sec \theta - \frac{y}{b} \tan \theta = k. \dots\dots(2)$

Now find the area of the triangle formed by (2) and the lines $y = \pm \frac{b}{a} x$.

Or. Let the equation of a hyperbola referred to its asymptotes as axes be $xy = c^2$, then any point on it is $(ct, \frac{c}{t})$.

Chord of contact of tangents drawn from $(ct, \frac{c}{t})$ to $2xy = k$ is

$$\frac{c}{t}x + cty = k. \dots\dots(3)$$

Now find the area of the triangle formed by (3) and the axes, etc.

17. Any point on $\frac{x}{a} - \frac{y}{b} = 0$ is (at, bt) . Its polar w. r. t. $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x}{a}t - \frac{y}{b}t = 1$. Hence etc.

18. If e, e' be respectively the eccentricities of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \text{and} \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1,$$

then $b^2 = a^2 (e^2 - 1)$ and $a^2 = b^2 (e'^2 - 1)$.

Hence, $(e^2 - 1)(e'^2 - 1) = 1$, etc.

19. The chord through 'a' and 'θ' is

$$\frac{x}{a} \cos \frac{1}{2} (\theta - \alpha) - \frac{y}{b} \sin \frac{1}{2} (\theta + \alpha) = \cos \frac{1}{2} (\theta + \alpha), \quad \dots\dots (1)$$

and that through 'β' and 'θ' is

$$\frac{x}{a} \cos \frac{1}{2} (\theta - \beta) - \frac{y}{b} \sin \frac{1}{2} (\theta + \beta) = \cos \frac{1}{2} (\theta + \beta). \quad \dots\dots (2)$$

If the lines (1) and (2) meet $\frac{x}{a} = \frac{y}{b}$ in (x_1, y_1) and (x_2, y_2) , then the portion intercepted on this asymptote

is $\frac{x_1 - x_2}{a} \sqrt{a^2 + b^2}$.

Now, find x_1 and x_2 and show that the portion intercepted is independent of θ , etc.

Or. Take the hyperbola as $xy = c^2$, the fixed points as $(ct_1, \frac{c}{t_1})$ and $(ct_2, \frac{c}{t_2})$ and the variable point as $(ct, \frac{c}{t})$ etc.

20. Eliminating θ , we have

$$\alpha - \beta = \tan^{-1} \frac{x}{a} - \tan^{-1} \frac{y}{b} = \tan^{-1} \frac{bx - ay}{ab + xy};$$

∴ the given equations represent the hyperbola

$$xy - (bx - ay) \cot (\alpha - \beta) + ab = 0, \text{ etc.}$$

21. The circle on the line joining the foci of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

as diameter is $x^2 + y^2 = a^2 e^2 = b^2 e'^2$,

$$\therefore \frac{1}{e^2} + \frac{1}{e'^2} = 1, \text{ etc.}$$

22. *Correction* : The hyperbola is rectangular.

The tangents at $(a\sqrt{2}, \pm a)$ on the rect. hyperbola $x^2 - y^2 = a^2$ are $\sqrt{2}x \mp y = a$ which pass through $(0, \mp a)$, the vertices of the conjugate hyperbola.

23. The conjugate hyperbola of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$. The polar of $(a \tan \theta, b \sec \theta)$ on the 2nd hyperbola w. r. t. the 1st is

$$\frac{x}{a} \tan \theta - \frac{y}{b} \sec \theta = 1$$

which is a tangent to the conjugate hyperbola at $(-a \tan \theta, -b \sec \theta)$.

24. Take the two perpendicular lines as axes. If the other st. line cuts off lengths h, k from the axes, then $hk = \text{constant}$ for all values of h and k . The centroid of the triangle formed is given by $x = \frac{1}{3} h, y = \frac{1}{3} k$.

Hence, the required locus is $xy = \frac{1}{9} hk = \text{constant}$.

25. The tangent at $(a \tan \phi, b \sec \phi)$ on the conjugate hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ is $\frac{x}{a} \tan \phi - \frac{y}{b} \sec \phi = -1$ which is a chord of $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with $(a \tan \phi, b \sec \phi)$ as its mid-point.

26. Ex. 16 repeated by mistake.

27. If (h, k) be a focus and $lx + my = 1$ be the corresponding directrix, then the equation of the hyperbola is

$$(x-h)^2 + (y-k)^2 = 2 \frac{(lx + my - 1)^2}{l^2 + m^2}. \quad \dots (1)$$

Compare (1) with $xy = c^2$, and find out h, k, l, m .

28. The equation $x^2 - y^2 = a^2$ when referred to the vertex $(a, 0)$ as the origin becomes $x^2 - y^2 + 2ax = 0$. $\dots (1)$

Now, find the equation of the st. lines joining the origin to the points of intersection of (1) and $y = mx + c$. If these lines contain a rt. \angle , we find that $m = 0$. Hence, etc.

29. The equation of a series of rect. hyperbolas having the same asymptotes is $x^2 - y^2 = \lambda$, where λ is a parameter.

The lines $y = mx$ and $y = m'x$ form a pair of conjugate diameters, if $mm' = -1$, which is independent of λ . Hence, etc.

30. A chord of the given hyperbola in terms of its mid. point (x_1, y_1) is

$$xy_1 + x_1y - 2c^2 = 2x_1y_1 - 2c^2, \text{ or } xy_1 + x_1y = x_1y_1.$$

If this chord be parallel to the diameter $y = mx$, then

$$m = -\frac{y_1}{x_1}, \text{ or } y_1 + mx_1 = 0.$$

Hence, the locus of (x_1, y_1) , i.e., the conjugate diameter is
 $y + mx = 0.$

31. The tangent $\frac{x}{a} - \frac{y}{b} \sin \phi = \cos \phi$ meets the asymptotes $\frac{x}{a} - \frac{y}{b} = 0$ and $\frac{x}{a} + \frac{y}{b} = 0$ respectively in the points $\left(\frac{a \cos \phi}{1 - \sin \phi}, \frac{b \cos \phi}{1 - \sin \phi} \right)$ and $\left(\frac{a \cos \phi}{1 + \sin \phi}, \frac{b \cos \phi}{1 + \sin \phi} \right)$.
Hence, etc.

32. The polar of (x_1, y_1) w. r. t. the hyperbola is
 $xy_1 + x_1y = 2c^2.$

If this line is $x - a = 0$, then $x_1 = 0$ and $y_1 = \frac{2c^2}{a}.$

If the two lines are conjugate, then (x_1, y_1) i.e., $(0, \frac{2c^2}{a})$ lies on $y - \beta = 0$, etc.

33. The normal at 't' is

$$xt - \frac{y}{t} = c \left(t^2 - \frac{1}{t^2} \right), \dots\dots(1)$$

and the chord joining 't' and 't'' is

$$x + ytt' = c(t + t'). \dots\dots(2)$$

Now (1) and (2) are identical. Hence, etc.

34. See Note, solved Ex. 3, page 88.

35. Let the rect. hyperbola $xy = c^2$ meet the circle

$$x^2 + y^2 + 2gx + 2fy + k = 0$$

in the four points 't', $r = 1, 2, 3, 4$:

then $CP^2 + CQ^2 + CR^2 + CS^2 = c^2 \sum t_r^2 + c^2 \Sigma \frac{1}{t_r^2}$

$$=c^2 \left(\frac{4g^2}{c^2} - \frac{2k}{c^2} \right) + c^2 \left(\frac{4f^2}{c^2} - \frac{2k}{c^2} \right) \left[\text{see Ex. 3, page 87.} \right]$$

$$=4 (g^2 + f^2 - k).$$

36. The st. lines joining any point Q ' t ' to P ' t_1 ' and P ' $-t_1$ ' are respectively

$$x + ytt_1 = c(t + t_1) \text{ and } x - ytt_1 = c(t - t_1),$$

which make equal angles with either axis. Hence etc.

37. If (x_1, y_1) is the pole of the normal at ' t ', then

$$xy_1 + x_1y = 2c^2 \text{ and } xt - y/t = c(t^2 - 1/t^2)$$

are identical. Compare and eliminate t .

38. If (x_1, y_1) be the mid-point of a chord of length $2d$ and making an angle θ with the x -axis, then the extremities of the chord are $(x_1 + d \cos \theta, y_1 + d \sin \theta)$ and $(x_1 - d \cos \theta, y_1 - d \sin \theta)$. Write the conditions that these lie on $xy = c^2$ and eliminate θ .

39. Use the result of Ex. 5, page 88.

Or. The normal at ' t ' on $xy = c^2$ is $xt - y/t = c(t^2 - 1/t^2)$.

This passes through $(cp, c/p)$ on the hyperbola, if

$$pt^4 - p^2t^3 + t - p = 0, \text{ or } (t - p)(pt^3 + 1) = 0.$$

Hence, the points P, Q, R are the points ' t_r ', $r = 1, 2, 3$, where t_1, t_2, t_3 are the roots of $pt^3 + 1 = 0$. Now, the centroid of $\triangle PQR$ is given by $3x = c \Sigma t_1, 3y = c \Sigma \frac{1}{t_1}$, etc.

40. If the normals at ' t_r ', $r = 1, 2, 3, 4$ meet in a point, then $t_1t_2t_3t_4 = -1$. If the circle through ' t_1 ', ' t_2 ', ' t_3 ' meets the hyperbola in ' t ', then $tt_1t_2t_3 = 1$, see Ex. 3, page 87 ;
 $\therefore t = -t_4$.

41. The normal at ' θ ' is $x \sec \theta - y \tan \theta = a$(1)

If (x_1, y_1) be the mid. point of this chord, then (1) is identical with $xx_1 - yy_1 = x_1^2 - y_1^2$(2)

Compare (1) and (2) and eliminate θ .

42. Let P, Q, R be respectively the points ' t_1 ', ' t_2 ' and ' t_3 ' on the hyperbola $xy = c^2$, then the st. line through P perpendi-

cular to QR is $y - \frac{c}{t_1} = t_2 t_3 (x - ct_1)$,

or $y + ct_1 t_2 t_3 = t_2 t_3 (x + \frac{c}{t_1 t_2 t_3})$,

and the st. line through Q perpendicular to RP is

$$y + ct_1 t_2 t_3 = t_3 t_1 (x + \frac{c}{t_1 t_2 t_3}), \text{ etc.}$$

43. Substitute $mx + 2c\sqrt{-m}$ for y in the equation $xy = c^2$.

44. The chord joining ' t_1 ' and ' t_2 ' on $xy = c^2$ is perpendicular to the chord joining ' t_3 ' and ' t_4 ' if $t_1 t_2 t_3 t_4 = -1$,

where $\tan \alpha = \frac{1}{t_1^2}$, etc.

46. Take asymptotes as axes. The chord of $xy = c^2$ with its mid. point as (x_1, y_1) is $xy_1 + x_1 y = 2x_1 y_1$, or $\frac{x}{2x_1} + \frac{y}{2y_1} = 1$.

The intercepts on the axes are $2x_1, 2y_1$. Thus, the mid point of a chord PQ is also the mid. point of the portion $P'Q'$ intercepted between the asymptotes.

47. If the hyperbola meets the first chord in ' t_1 ' and ' t_2 ', and 2nd in ' t_3 ' and ' t_4 ', then t_1 and t_2 are the roots of

$$ct^2 \cos \alpha - pt + c \sin \alpha = 0,$$

and t_3, t_4 are the roots of $t^2 \sin \alpha - \frac{c \cos 2\alpha}{p} t - \cos \alpha = 0$,

$\therefore t_1 t_2 = \tan \alpha$, and $t_3 t_4 = -\cot \alpha$. Hence, etc.

Miscellaneous Exercise

1. Any tangent to the parabola is $y = mx + \frac{2a}{m}$. Now, find m such that it also touches the circle.

2. Points on parabola whose abscissae are in the ratio $m^2 : 1$ are $(am^2 t^2, 2amt)$ and $(at^2, 2at)$.

Eliminating t between $mt y = x + am^2 t^2$ and $ty = x + at^2$, the required locus is $y^2 = (m^{1/2} + m^{-1/2})^2 ax$.

3. The pole of $ty = x + at^2$ w. r. t. $y^2 = -4ax$ is $(at^2, -2at)$ which evidently lies on $y^2 = 4ax$.

4. If the points of contact of the tangents drawn from (x_1, y_1) be ' t_1 ' and ' t_2 ', then $x_1 = at_1 t_2$ and $y_1 = a(t_1 + t_2)$.

Now, length of the chord of contact

$$= a [(t_1^2 - t_2^2)^2 + 4(t_1 - t_2)^2]^{\frac{1}{2}} \text{ etc.}$$

5. If P be the point (x_1, y_1) , then $\frac{y_1^2 - 4ax_1}{\sqrt{y_1^2 + 4a^2}} = k$, etc.

6. The polar of (x_1, y_1) w. r. t. the parabola is $yy_1 = 2a(x + x_1)$.

The equation of the lines joining the vertex to the points of intersection of the parabola and the polar of (x_1, y_1) is

$$y^2 - 4ax (yy_1 - 2ax)/(2ax_1) = 0.$$

7. The normals at ' t_1 ' and ' t_2 ' intersect at rt. \angle s, if $t_1 t_2 = -1$.

The chord joining ' t_1 ' and ' t_2 ' is

$$y(t_1 + t_2) = 2(x + at_1 t_2). \text{ Hence etc.}$$

8. The tangents at C ' t_1 ' and D ' t_2 ' meet in $P(x, y)$, where

$$x = at_1 t_2, \quad y = a(t_1 + t_2).$$

The tangent at the vertex meets the tangents at C and D respectively in the points $Q(0, at_1)$ and $R(0, at_2)$;

\therefore area of $\triangle PQR = \frac{1}{2} at_1 t_2 (at_1 - at_2)$, etc.

9. The tangent at ' t ' meets the directrix in $(-a, at - a/t)$; \therefore the mid-point of the tangent intercepted between the directrix and the parabola is given by

$$2x = at^2 - a, \quad 2y = 3at - a/t,$$

Now, eliminate t .

10. The tangents at C ' t_1 ' and D ' t_2 ' meet in $P(x, y)$, where $x = at_1 t_2$, $y = a(t_1 + t_2)$. The directrix meets the tangents

at C and D respectively in the points $M(-a, at_1 - \frac{a}{t_1})$ and $N(-a, at_2 - \frac{a}{t_2})$. Now, $d = MN = a(t_1 - t_2) - a(\frac{1}{t_1} - \frac{1}{t_2})$, etc.

11. If the points ' ϕ ' and ' $\phi + \pi/2$ ' lie on $lx + my = n$, then $al \cos \phi + bm \sin \phi = n$, and $al \sin \phi - bm \cos \phi = -n$.

Now, eliminate ϕ .

12. Let S, S' be the foci of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$(1)

If the foci H, H' of a concentric ellipse lie on (1), then $SHS'H'$ is obviously a parallelogram and $SH + S'H = SH + SH'$. Thus, the major axis of the 2nd ellipse is also $2a$. If the angle between the two ellipses is θ , then

$$SH^2 = CS^2 + CH^2 - 2 CS \cdot CH \cos \theta, \text{ where } C \text{ is the centre}$$

$$= a^2 e^2 + a^2 e'^2 - 2a^2 ee' \cos \theta, \text{(1)}$$

$$\text{and } SH'^2 = a^2 e^2 + a^2 e'^2 + 2a^2 ee' \cos \theta. \text{(2)}$$

$$\text{Hence, } 4a^2 = (SH + SH')^2 = SH^2 + SH'^2 + 2SH \cdot SH'. \text{(3)}$$

Now, find out $\cos \theta$ from (1), (2) and (3).

13. The st. lines joining the centre to the points of contact of tangents from (x_1, y_1) are $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \left(\frac{xx_1}{a^2} + \frac{yy_1}{b^2}\right)^2$.

Now these contain a rt. \angle , if etc.

14. Tangents at ' α ' and ' β ' ($\beta > \alpha$) on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ meet in the point $\left[a \frac{\cos \frac{1}{2}(\beta + \alpha)}{\cos \frac{1}{2}(\beta - \alpha)}, b \frac{\sin \frac{1}{2}(\beta + \alpha)}{\cos \frac{1}{2}(\beta - \alpha)} \right]$

which lies on $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 4$, if $\cos \frac{1}{2}(\beta - \alpha) = \frac{1}{2}$,

i.e., if $\beta = \alpha + \frac{2\pi}{3}$.

Now, eliminate α between the equations of normals at ' α ' and ' $\alpha + \frac{2\pi}{3}$ ',

15. If (x_1, y_1) be the pole of $x \cos \alpha + y \sin \alpha = c$ (1) w. r. t. the ellipse, then (1) and $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ are identical.

Compare and eliminate α .

16. The chord of the ellipse with its mid. point as (x_1, y_1) is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$(1)

If (1) be the chord of contact of tangents drawn from $(\sqrt{a^2 + b^2} \cos \alpha, \sqrt{a^2 + b^2} \sin \alpha)$, then etc.

17. The chord whose mid-point is (x_1, y_1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}. \quad \dots\dots\dots(1)$$

Compare (1) with $ax \sec \phi - by \operatorname{cosec} \phi = a^2 - b^2$ and eliminate ϕ .

18. Write the conditions that the points

$(x_1 + c \cos \theta, y_1 + c \sin \theta)$ and $(x_1 - c \cos \theta, y_1 - c \sin \theta)$ lie on the ellipse and eliminate θ .

19. Let the four tangents be

$$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = 1, \quad \dots\dots\dots(1)$$

$$\frac{x}{a} \cos \alpha + \frac{y}{b} \sin \alpha = -1, \quad \dots\dots\dots(2)$$

$$\frac{x}{a} \cos \beta + \frac{y}{b} \sin \beta = 1, \quad \dots\dots\dots(3)$$

and $\frac{x}{a} \cos \beta + \frac{y}{b} \sin \beta = -1, \quad \dots\dots\dots(4)$

then (1) and (3) meet on $x = h$, if $h = a \frac{\cos \frac{1}{2}(\alpha + \beta)}{\cos \frac{1}{2}(\alpha - \beta)}$. $\dots\dots\dots(5)$

The point of intersection of (1) and (4) is given by

$$\frac{x}{a} = \frac{\sin \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)}, \quad \frac{y}{b} = \frac{\cos \frac{1}{2}(\alpha + \beta)}{\sin \frac{1}{2}(\alpha - \beta)}. \quad \dots\dots\dots(6)$$

Eliminate α, β between (5) and (6).

20. The equation of st. lines joining the centre to the intersections of the given line with the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{(y - mx)^2}{\frac{1}{2}(a^2 m^2 + b^2)}.$$

The product of the slopes of these lines = etc.

21. Let P, D, P', D' be respectively the points

$$' \phi', ' \phi + \pi/2', ' \phi + \pi' \text{ and } ' \phi + 3\pi/2'$$

and let Q be $(c \cos \alpha, c \sin \alpha)$, then

$$PQ^2 = (c \cos \alpha - a \cos \phi)^2 + (c \sin \alpha - b \sin \phi)^2, \text{ etc.}$$

22. If the normal at (x_1, y_1) on the ellipse passes through (h, k) , then $\frac{a^2 h}{x_1} - \frac{b^2 k}{y_1} = a^2 - b^2$, etc.

23. The series of hyperbolas is given by

$$\frac{x^2}{a^2} - \frac{y^2}{\lambda^2} = 1, \quad \dots\dots\dots(1)$$

where λ is a parameter. If P be the point (x_1, y_1) on (1), then

$$y_1 = \text{distance from the asymptote } \frac{x}{a} - \frac{y}{\lambda} = 0, \quad \text{say}$$

$$= \frac{\lambda x_1 - a y_1}{\sqrt{\lambda^2 + a^2}}. \quad \dots\dots\dots(2)$$

Eliminate λ between (1) and (2).

$$24. \quad x = \lambda \text{ meets } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ in } P \left(\lambda, \frac{b}{a} \sqrt{\lambda^2 - a^2} \right)$$

$$\text{and } \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1 \text{ in } Q \left(\lambda, \frac{b}{a} \sqrt{\lambda^2 + a^2} \right)$$

Now, write the equations of the tangents at P and Q to their respective curves and eliminate λ .

25. Let P, Q, R be respectively the points ' t_1 ', ' t_2 ' and ' t_3 ', then QR is parallel to $y + lx = 0$, if $t_2 t_3 = -\frac{1}{l}$.

Similarly,

$$t_3 t_1 = -\frac{1}{m}, \quad t_1 t_2 = -\frac{1}{n}; \quad \therefore t_1 = \sqrt{\frac{l}{mn}}, \text{ etc.}$$

$$\text{Area of the } \triangle PQR = \frac{c^2}{2t_1 t_2 t_3} (t_2 - t_3)(t_3 - t_1)(t_1 - t_2), \text{ etc.}$$

26. $x^2 - y^2 = \phi$, and $2xy = \psi$. Hence etc.

27. The tangent at $P(\phi)$ meets the asymptote $\frac{x}{a} = \frac{y}{b}$ in

$$Q \left(\frac{a}{\sec \phi - \tan \phi}, \frac{b}{\sec \phi - \tan \phi} \right)$$

and the asymptote

$$\frac{x}{a} + \frac{y}{b} = 0 \text{ in } R \left(\frac{a}{\sec \phi + \tan \phi}, \frac{-b}{\sec \phi + \tan \phi} \right).$$

If the circle through C, P, Q is

$$x^2 + y^2 + 2gx + 2fy = 0,$$

then $-2ag = (a^2 + b^2) \sec \phi$ and $-2bf = (a^2 + b^2) \tan \phi$.

Now eliminate ϕ .

28. Proceed as in Ex. 22 above.

Exercise 7

8. Change the origin to (c, c') and eliminate t .

This gives

$$(ay' - a'x)^2 = (ab' - a'b)(b'x - by),$$

which is a parabola. Hence etc.

9. Find the equation of the conic when referred to parallel axes through the centre. Now apply § 7.41.

10. Solve the equation in Ex. 9 above for $\frac{1}{r^2}$.

11. Axes are the lines through the foci and the centre. Now see solved Ex. 3, page 119.

12. Take the lines $x - 2y + 1 = 0$ and $4x + 2y - 3 = 0$, which are at rt. \angle s, as new axes.

13. Referred to parallel axes through the centre, the given equation becomes

$$ax^2 + 2hxy + by^2 + \Delta / (ab - h^2) = 0. \quad \dots\dots\dots (1)$$

If the axes be now rotated so as to coincide with the asymptotes (which are at rt. \angle s), the equation (1) is transformed into

$$2Hxy + \Delta / (ab - h^2) = 0. \quad \dots\dots\dots (2)$$

By invariants, $h^2 - ab = H^2$, i. e., $H = \pm \sqrt{h^2 - ab}$.

14. Equation of the hyperbola is

$$(3x - y + 7)(4x + 3y + 1) = k, \quad \dots\dots\dots (1)$$

where k is such that $(0, 0)$ lies on (1).

Now proceed as in solved Ex. 2, page 116.

Exercise 8

5. In $\frac{\sin(\theta_1 - \theta_2)}{r} = \frac{\sin(\theta_1 - \theta)}{r_2} + \frac{\sin(\theta - \theta_2)}{r_1}$, put $\theta = \frac{1}{2}(\theta_1 + \theta_2)$ and find out r .

6. The vectorial angles θ_1, θ_2 of the points of intersection are given by $1 + e \cos \theta = l(a \cos \theta + b \sin \theta)$,

or $[b^2 l^2 + (la - e)^2] \cos^2 \theta - 2(la - e) \cos \theta + 1 - b^2 l^2 = 0$;

$\therefore \cos \theta_1 \cos \theta_2 = \frac{1 - b^2 l^2}{b^2 l^2 + (la - e)^2}$.

Similarly, find out $\sin \theta_1 \sin \theta_2$ and use $\theta_1 \sim \theta_2 = \frac{\pi}{2}$.

7. Eliminating r , we get

$$2c \cos \theta \cdot (a \cos \theta + b \sin \theta) = 1 = \cos^2 \theta + \sin^2 \theta,$$

or

$$\tan^2 \theta - 2bc \tan \theta + 1 - 2ac = 0.$$

Now, write the condition that this equation has equal roots.

8. The circle $r^2 - 2r(A \cos \theta + B \sin \theta) + C = 0$ (1) touches the initial line, if $r^2 - 2rA + C = 0$ has equal roots, i.e., if $C = A^2$. Since the point of contact is $(0, c)$, $A = c$.

Equation (1) now becomes $r^2 - 2cr \cos \theta + c^2 = 2Br \sin \theta$.

Now, write the condition that it passes through (r_1, θ_1) and eliminate B .

9. The equation of the given circle referred to O as the pole and the diameter through O as the initial line is

$$r^2 - 2a r \cos \theta + b = 0, \text{ where } a, b \text{ are constants.}$$

If P be the point (r_1, θ_1) , then the coordinates of Q are given by $r = k r_1, \theta = \theta_1$. Since P lies on the circle,

$$r_1^2 - 2a r_1 \cos \theta_1 + b = 0;$$

\therefore the locus of Q is $\frac{k^2}{r^2} - 2a \frac{k}{r} \cos \theta + b = 0$, etc.

10. The equation of the circle referred to O as the pole and the diameter through O as the initial line is $r = 2a \cos \theta$,

where a is the radius. Let the points A, B, C be respectively $(2a \cos \alpha, \alpha)$, $(2a \cos \beta, \beta)$ and $(2a \cos \gamma, \gamma)$, then the equations of BC, CA, AB are respectively

$$2a \cos \beta \cos \gamma = r \cos (\theta - \beta - \gamma), \quad \dots\dots (1)$$

$$2a \cos \gamma \cos \alpha = r \cos (\theta - \gamma - \alpha), \quad \dots\dots (2)$$

and $2a \cos \alpha \cos \beta = r \cos (\theta - \alpha - \beta). \quad \dots\dots (3)$

Hence, the coordinates of the feet of the perpendiculars from O on the lines (1), (2), (3) are respectively

$$(2a \cos \beta \cos \gamma, \beta + \gamma), (2a \cos \gamma \cos \alpha, \gamma + \alpha)$$

and $(2a \cos \alpha \cos \beta, \alpha + \beta)$ which obviously lie on the st. line

$$2a \cos \alpha \cos \beta \cos \gamma = r \cos (\theta - \alpha - \beta - \gamma).$$

11. The centres of similitude divide the line joining the centres internally and externally in the ratio of the radii and are easily seen to be $\left(\frac{2ab}{a+b}, 0\right)$ and $(0, 0)$.

Now, proceed as in § 8.31.

12. Write the equation of a st. line perpendicular to the st. line joining (r_1, θ_1) and (c, α) . Now, use the condition that this line passes through (r_1, θ_1) .

14. P, P' the extremities of a focal chord PSP' of

$$\frac{l}{r} = 1 + e \cos \theta$$

are (r_1, α) and $(r_2, \pi + \alpha)$ respectively, where

$$\frac{l}{r_1} = 1 + e \cos \alpha, \quad \text{and} \quad \frac{l}{r_2} = 1 + e \cos (\pi + \alpha) = 1 - e \cos \alpha.$$

Now add to get $\frac{1}{r_1} + \frac{1}{r_2}.$

15. If PSP', QSQ' be two perpendicular focal chords of

$$\frac{l}{r} = 1 + \sqrt{2} \cos \theta, \quad \text{then } P, P', Q, Q' \text{ can be taken resp. :}$$

as $(r_1, \theta_1), (r_2, \theta_1 + \pi), \left(r_3, \theta_1 + \frac{\pi}{2}\right)$ and $\left(r_4, \theta_1 + \frac{3\pi}{2}\right).$

$$\text{Now, } PP' = r_1 + r_2 = \frac{2l}{1 - 2 \cos^2 \theta_1}, \quad QQ' = -\frac{2l}{1 - 2 \sin^2 \theta_1}.$$

Hence, etc.

16. Let a be the vect : \angle of the extremity P of a focal chord PSQ of the conic $\frac{l}{r} = 1 + e \cos \theta$, then

$$SP = \frac{l}{1 + e \cos a}, \quad SQ = \frac{l}{1 - e \cos a}.$$

If the mid. point R of PQ be (ρ, a) , then

$$\begin{aligned} \rho &= SP - PR = SP - \frac{1}{2} (SP + SQ) = \frac{1}{2} (SP - SQ) \\ &= -\frac{l e \cos a}{1 - e^2 \cos^2 a}, \end{aligned}$$

whence the locus of R is $r + \frac{l e \cos \theta}{1 - e^2 \cos^2 \theta} = 0$. Now, change to cartesian coordinates to discuss the nature of the locus.

17. If the ellipse be $\frac{l}{r} = 1 + e \cos \theta$, then

$$\frac{1}{SP} + \frac{1}{SQ} = -\frac{2}{l}, \quad \text{or} \quad \frac{SP}{SQ} = -\frac{2}{l} SP - 1.$$

Similarly, $\frac{HP}{HR} = -\frac{2}{l} HP - 1$. Now add, etc.

18. Let the conic and the circle be respectively

$$\frac{l}{r} = 1 + e \cos \theta \quad \text{and} \quad r = d \cos (\theta - \gamma), \quad \text{then}$$

r_1, r_2, r_3, r_4 are the roots of the equation obtained by eliminating θ between these equations, etc.

19. The tangent at any point ' a ' on $\frac{l}{r} = 1 + e \cos \theta$ is

$$\frac{l}{r} = e \cos \theta + \cos (\theta - a).$$

It meets the directrix $\frac{l}{r} = e \cos \theta$, where

$$\cos (\theta - a) = 0, \text{ etc.}$$

20. The tangents at P ' $\alpha - \beta$ ' and Q ' $\alpha + \beta$ ' meet in T ' θ ', where $\cos (\theta - \alpha + \beta) = \cos (\theta - \alpha - \beta)$, i.e., $\theta = \alpha$.

Also the st. line PQ , i. e., $\frac{l}{r} = e \cos \theta + \sec \beta \cos (\theta - \alpha)$

meets $\frac{l}{r} = e \cos \theta$, where $\cos (\theta - \alpha) = 0$. Hence etc.

21. The tangents at P ' $\alpha - \beta$ ' and Q ' $\alpha + \beta$ ' meet in T ' α ' [see Ex. 20 above] ;

$\therefore T$ is the point $\left(\frac{l}{e \cos \alpha + \cos \beta}, \alpha \right)$.

(i) If the conic be a parabola, $e = 1$.

$$\begin{aligned} \text{Now, } SP \cdot SQ &= \frac{l}{1 + \cos (\alpha - \beta)} \cdot \frac{l}{1 + \cos (\alpha + \beta)} \\ &= \frac{l^2}{4} \sec^2 \frac{\alpha - \beta}{2} \sec^2 \frac{\alpha + \beta}{2}, \end{aligned}$$

$$\text{and } ST^2 = \left(\frac{l}{\cos \alpha + \cos \beta} \right)^2 = \frac{l^2}{4} \sec^2 \frac{\alpha + \beta}{2} \sec^2 \frac{\alpha - \beta}{2}.$$

Hence, etc.

$$(ii) \quad \frac{l^2}{SP \cdot SQ} = [1 + e \cos (\alpha - \beta)] [1 + e \cos (\alpha + \beta)],$$

$$\text{and } \frac{l^2}{ST^2} = (e \cos \alpha + \cos \beta)^2 ;$$

$$\begin{aligned} \therefore \frac{l^2}{SP \cdot SQ} - \frac{l^2}{ST^2} &= 1 - \cos^2 \beta + e^2 [\cos (\alpha - \beta) \cos (\alpha + \beta) - \cos^2 \alpha] \\ &= (1 - e^2) \sin^2 \beta. \quad \text{Hence, etc.} \end{aligned}$$

22. If the two conics touch each other at the point ' β ',

$$\text{then } \frac{l_1}{r} = e_1 \cos \theta + \cos (\theta - \beta),$$

$$\text{and } \frac{l_2}{r} = e_2 \cos \theta + \cos (\theta - \beta) \quad \text{are identical.}$$

Now compare and eliminate β .

23. Let P, Q, R be respy : the points ' $\alpha - \beta$ ', ' γ ' and ' $\alpha + \beta$ ', then the tangent at Q is $\frac{l}{r} = e \cos \theta + \cos (\theta - \gamma)$.
(1)

Since $M(l, \alpha - \beta)$ and $N(l, \alpha + \beta)$ lie on (1),

$$1 = e \cos (\alpha - \beta) + \cos (\alpha - \beta - \gamma) \\ = (e + \cos \gamma) \cos (\alpha - \beta) + \sin \gamma \sin (\alpha - \beta),$$

and $1 = e \cos (\alpha + \beta) + \cos (\alpha + \beta - \gamma)$
 $= (e + \cos \gamma) \cos (\alpha + \beta) + \sin \gamma \sin (\alpha + \beta) ;$

$$\therefore e + \cos \gamma = \cos \alpha \sec \beta \text{ and } \sin \gamma = \sin \alpha \sec \beta. \quad \dots (2)$$

Now simplify the equation of PR by using (2), etc.

24. The common chords of $\frac{l}{r} = 1 + e \cos \theta$

and $\frac{l'}{r} = 1 + e' \cos (\theta - \gamma)$ are

$$\frac{l}{r} - e \cos \theta = \pm \left[\frac{l'}{r} - e' \cos (\theta - \gamma) \right].$$

Now the chord $\frac{l-l'}{r} = e \cos \theta - e' \cos (\theta - \gamma)$ always

touches the conic $\frac{l-l'}{r} = -e' + e \cos \theta$, etc.

25. The vertices of the triangle are

$$\left(a \sec \beta /_2 \sec \gamma /_2, \frac{\beta + \gamma}{2} \right),$$

$\left(a \sec \gamma /_2 \sec \alpha /_2, \frac{\gamma + \alpha}{2} \right)$ and $\left(a \sec \alpha /_2 \sec \beta /_2, \frac{\alpha + \beta}{2} \right)$ which obviously lie on the circle

$$r = a \sec \alpha /_2 \sec \beta /_2 \sec \gamma /_2 \cos \left(\theta - \frac{\alpha + \beta + \gamma}{2} \right).$$

26. Let the common tangent touch $\frac{l}{r} = 1 + e \cos \theta$ at ' α '

and $\frac{l}{r} = 1 + e \sin \theta$ at ' β ', then

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha) \quad \text{and} \quad \frac{l}{r} = e \sin \theta + \cos (\theta - \beta)$$

are identical. Compare and show that $\frac{\alpha + \beta}{2} = \frac{\pi}{4}$.

Hence find out $\frac{\alpha - \beta}{2}$.

27. If the conics $\frac{l}{r} = 1 + e \cos \theta$ and $\frac{l'}{r} = 1 + e \cos (\theta - \gamma)$

touch each other at 'a', then

$$\frac{l}{r} = e \cos \theta + \cos (\theta - \alpha) \quad \text{and} \quad \frac{l'}{r} = e \cos (\theta - \gamma) + \cos (\theta - \alpha)$$

are identical. Compare and eliminate γ .

29. If the tangent at 'a' on $\frac{ae_1}{r} = 1 + e_1 \cos \theta$

meets $\frac{ae_2}{r} = 1 + e_2 \cos \theta$ in ' β ',

then $e_1 (1 + e_2 \cos \beta) = e_2 [\cos (\beta - \alpha) + e_1 \cos \beta]$, etc.

30. If the tangent at 'a' on $\frac{2a}{r} = 1 + \cos \theta$

is perpendicular to the tangent at ' β ' on

$$\frac{2b}{r} = 1 + \cos (\theta - \gamma),$$

then $(1 + \cos \alpha) (\cos \beta + \cos \gamma) + \sin \alpha (\sin \beta + \sin \gamma) = 0$,

or $\alpha = \pi + \beta + \gamma$(1)

Now eliminate α, β from the equations of the tangents by using (1).

31. The normal at ' μ ' on the given parabola is

$$\frac{2a}{r} \cdot \frac{\sin \mu}{1 - \cos \mu} = \sin \theta - \sin (\theta - \mu),$$

or $t^3 \sin \theta + \left(\frac{a}{r} - \cos \theta \right) t^2 + \frac{a}{r} = 0$, where $t = \tan \mu/2$.

If this passes through a point whose vectorial \angle is δ , then

$$t^3 \sin \delta + \left(\frac{a}{r} - \cos \delta \right) t^2 + \frac{a}{r} = 0.$$

The roots of this cubic are $\tan \alpha/2$, $\tan \beta/2$ and $\tan \gamma/2$, etc.

32. The normal at ' μ ' on the conic passes through (ρ, ϕ) ,

if
$$\frac{l e \sin \mu}{1 + e \cos \mu} \cdot \frac{1}{\rho} = e \sin \phi + \sin (\phi - \mu),$$

or
$$(1 - e)^2 \rho t^4 \sin \phi + 2t^3 [l e + \rho (1 - e) \cos \phi] + 2t [l e + \rho (1 + e) \cos \phi] - \rho (1 + e)^2 \sin \phi = 0,$$

where
$$t = \tan \mu/2.$$

The roots of this biquadratic are $\tan \alpha/2$, $\tan \beta/2$, $\tan \gamma/2$ and $\tan \delta/2$.

33. The points of intersection of the conic and the normal at L ' $\pi/2$ ' are given by

$$\left(e l + \frac{l - r}{e} \right)^2 = e^2 r^2 \left[1 - \left(\frac{l - r}{e r} \right)^2 \right].$$

One root of this quadratic is l and other is the required length.

34. If the polar of T meets the conic in P and Q and if the chord PQ meets the directrix in K , then $KST = \pi/2$, where S is the focus (see Ex. 20, page 157). Thus K is a fixed point.

35. Let the points P, Q on the conic $\frac{l}{r} = 1 + e \cos \theta$ be respectively ' α ' and ' $\alpha + \pi/2$ ', then

$$\frac{l}{SP} = 1 + e \cos \alpha, \text{ and } \frac{l}{SQ} = 1 + e \sin \alpha, \text{ etc.}$$

36. Let P, P' be respectively the points ' $\alpha - \beta$ ' and ' $\alpha + \beta$ ' on the conic $\frac{l}{r} = 1 + e \cos \theta$, S being the pole, and let O be the point (r_1, θ_1) , then

$$\frac{l}{r_1} = e \cos \theta_1 + \sec \beta \cos (\theta_1 - \alpha).$$

Hence $\frac{\cos(\theta_1 - \alpha)}{\cos \beta} = \frac{l}{r_1} - e \cos \theta_1 = \text{constant, etc.}$

37. Suppose the conic is the ellipse $\frac{l}{r} = 1 + e \cos \theta$

referred to S as the focus. Let S' be the other focus and let the vectorial \angle_s of P (on the conic) referred to S, S' be respectively θ, θ' , then

$$S'P + SP = 2a, S'P \cos \theta' - SP \cos \theta = SS' = 2ae, \dots\dots(1)$$

where $2a$ is the major axis of the ellipse.

$$\text{From equations (1), } \frac{2a(\cos \theta' - e)}{\cos \theta' + \cos \theta} = SP = \frac{a(1 - e^2)}{1 + e \cos \theta}$$

or $\cos \theta = \frac{(1 + e^2) \cos \theta' - 2e}{(1 + e^2) - 2e \cos \theta'}, \text{ etc.}$

Similarly the case when the conic is a hyperbola.

38. Let the equations of the ellipse and the hyperbola be respectively $\frac{l}{r} = 1 + e \cos \theta, \dots\dots(1)$

$$\frac{l'}{r} = 1 - e' \cos(\theta - \gamma) \quad [\text{nearer branch}], \dots\dots(2)$$

$$\text{and } \frac{l'}{r} = -1 - e' \cos(\theta - \gamma) \quad [\text{further branch}], \dots\dots(3)$$

Now eliminate θ from (1) and (2), and from (1) and (3)

to get $\frac{1}{r_1} + \frac{1}{r_2}$ and $\frac{1}{r_3} + \frac{1}{r_4}$.

Errata

Page	Line	For	Read
18	25, Ex. 7	—	Add (f, g) at the end
56	25	—	After 'coordinates of the' add 'centre of the'
60	14, Ex. 1	—	Add 'touches, after 'also'
62	16, Ex. 4	—	Add 'triangle' after 'centroid of'
72	5	$\overline{b^2}$	$\frac{y^2}{b^2}$
73	21, Ex. 1	$\frac{1}{a+b^2}$	$\frac{1}{a^2+b^2}$
87	last	$\frac{1}{1}$	$\frac{1}{t_1}$
90	15	x_2	x^2
91	15, Ex. 22	the hyperbola	a rectangular hyperbola
91	—	—	Delete Ex. 26
95	6, Ex. 2	loucs	locus
95	7	$m^{1/2} + m^{1/2}$	$m^{1/2} + m^{-1/2}$
97	19, Ex. 22	y	y^2
98	19, Ex. 27	b^2y	b^2y^2
98	22, Ex. 28	$+b^2kx$	$-b^2kx$
100	21	In	If
105	2	$\left(\frac{ax+hy+\lambda}{\sqrt{a^2+h^2}} \right)$	$\left(\frac{ax+hy+\lambda}{\sqrt{a^2+h^2}} \right)^2$
107	15, Ex.	$-16y^2$	$+16y^2$
121	25, Ex. 8	$(a^2+a'^2)^{\frac{2}{3}}$	$(a^2+a'^2)^{3/2}$
122	9, Ex. 13	$(h^2-ab)^{3/2}$	$(h^2-ab)^{3/2}$
124	3	15	14
126	10	makes	make
126	11	poistive	positive
128	21	$\theta-a$	$\theta-a$
128	24 & 25	—	$\frac{p}{r} = \cos (\theta-a)$

		(1)	(5)
129	3		
132	last	$r^2 \sin \theta$	$r \sin^2 \theta$
134	6	writers	writers
„	14	ccs α	$\cos \alpha$
136	3 from below	$e < 1$	$e > 1$
140	7	$A \cos (\alpha + \beta)$	$A \cos (\alpha - \beta)$
141	last	whree	where
144	12	if	if*
147	last	β_1	β
148	3	oos θ	$\cos \theta_1$
149	3 from below	l	e
149	last	enth	then
152	10	$\cos' \theta$	$\cos \theta$
152	last	\sim	\sim
156	11, Ex. 10	—	Add 'O' after 'point'
158	13, Ex. 25	$\frac{l}{r}$	$\frac{2a}{r}$
160	11, Ex. 38	$-\frac{1}{r_4}$	$+\frac{1}{r_4}$



